THE UNIVERSITY OF ALBERTA

# THE SIMPLEX METHOD AND ITS INTERPRETATIONS

BY

(C) JANG-TYE [illegible]

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF ARTS

DEPARTMENT OF ECONOMICS

EDMONTON, ALBERTA

FALL, 1970

THE UNIVERSITY OF ALBERTA

# THE SIMPLEX METHOD AND ITS INTERPRETATIONS

by

© JANG-TZE LIN

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF ARTS

DEPARTMENT OF ECONOMICS

EDMONTON, ALBERTA

FALL, 1970

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "The Simplex Method and its Interpretations" submitted by Jang-Tze Lin in partial fulfilment of the requirements for the degree of Master of Arts.

Date April 1, 1970

ABSTRACT

The aim of this thesis is to reorganize what I have learned and had in mind concerning the theory of linear programming. The focus of analysis is on the simplex method.

The first two chapters serve as a foundation of our subsequent discussion of the simplex method. Chapter 3 deals with the theoretical aspect of the simplex method, while Chapter 4 with the computational aspect. In Chapters 5 and 6, geometric and economic interpretation of the simplex method is discussed.

This thesis is not able to claim a contribution to the theory, however, all the theorems in the context are reinterpreted, and the proofs are modified. All the examples are either originated, or taken from exercises in the textbooks cited. I have solved and rearranged all the examples to suit the illustration of the theory.

My first access to the theory of linear programming was led by Dorfman, Samuelson, and Solow's "Linear Programing and Economic Analysis", however, my later study of Gass's "Linear Programming", and Hadley's "Linear Programming" has more influence on my thought and understanding of the theory. I hope this thesis will serve as a stepping-stone to my further study in this field.

ACKNOWLEDGEMENT

I should like to express my indebtedness to Professor B. von Hohenbalken and Professor A. Buse for their instruction. The order of contents was suggested by Professor von Hohenbalken. He also suggested some topics which I did not discuss in the first draft of my manuscript. Professor Buse corrected some theoretical and grammatical mistakes which I made in the second draft of my manuscript. Thanks also to my wife, Yoko, for her assistance in copying my manuscript and drawing diagrams.

CONTENTS

Page

LIST OF GRAPHS



LIST OF TABLES

# LIST OF NOTATION

In order to avoid confusion, we list the symbols which will appear in more than one chapter in our context. We shall also define all the symbols at the places of their first appearance. In this list, we shall not define the symbols which appear in only one chapter. We shall use as many well-recognized symbols as possible. However we also use our own symbols wherever necessary or convenient for our discussion. Note all vectors are defined as column vectors.

$A = (a_{ij})$ , a matrix with $a_{ij}$ as its elements $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$; $n > m$, $n$ and $m$ are finite numbers.

$A^* = (A,I)$ , an $m x(n + m)$ matrix where $I$ is an identity matrix of order $m$ .

$A_j = (a_{1j},a_{2j},\ldots,a_{mj})'$ , an $m \times 1$ column vector which is a column in matrix $A$ , or $A^*$ . $j$ ranges $1,2,\ldots,n$, or $1,2,\ldots,n$, $n + 1,\ldots,n + m$, depending on the matrix considered is $A$ or $A^*$ . Note in the discussion of the simplex method, $A_j$ is referred to the column vector not in the basis, while $A_i$ to the column vector in the basis. When the basis contains the first $m$ vectors, then $i \leqq m$ , $j > m$ .

$B_r$ an $m \times m$ matrix formed by $A_i$, $i = 1,2,\ldots,m$ . This matrix is called the basis. The subscript $r$ is the index number of basis.

$b = (b_1,b_2,\ldots,b_m)'$, an $m \times 1$ column vector

$f$ a number not greater than $k$ . See $r$ .

$k = \binom{n}{m}$ or $= \binom{n+m}{m}$ depending on whether the matrix considered is $A$ or $A^*$. $k$ represents the maximum possible number of bases.

$0$ a null vector whose dimension will be defined wherever it appears.

$r$ index number of bases which can range up to $k$. When $r = f$, it indicates the final basis, i.e. the optimal basis.

$S$ the feasible set.

$v = (v_1, v_2, \ldots, v_n)'$, an $n \times 1$ column vector, the values assigned to the variables in the primal objective function.

$v^* = \begin{bmatrix} V \\ 0 \end{bmatrix} = (v_1, v_2, \ldots, v_n, 0_{n+1}, \ldots, 0_{n+m})'$, an $(n + m) \times 1$ column vector.

$V_r = (v_1, v_2, \ldots, v_m)'$, an $m \times 1$ column vector, equivalent to $V$ when the basis $B_r$ contains the first $m$ columns in $A$ or $A^*$.

$W = b'y$, a scalar, the value of the dual objective function.

$W(y)$, $W$ is a function of $y$.

$x = (x_1, x_2, \ldots, x_n)'$, an $n \times 1$ column vector, the variables of the primal program.

$x^* = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})'$, an $(n + m) \times 1$ column vector when $A^*$ is involved.

$X_o^r = (x_{10}^r, x_{20}^r, \ldots, x_{mo}^r)'$, an $m \times 1$ column vector, the basic solution to primal program when the basis $B_r$ contains the first $m$ columns $A_i$, $i \leqq m$, in $A$ or $A^*$.

$X_j^r = (x_{1j}^r, x_{2j}^r, \ldots, x_{mj}^r)'$ , $j > m$ , an $m \times 1$ column vector, the equivalent combination when the basis $B_r$ contains the first $m$ columns $A_i$, $i \leqq m$ , in $A$ or $A^*$ . Special attention must be paid to the distinction between $X_o^r$ and $X_j^r$ .

$y = (y_1, y_2, \ldots, y_m)'$ , an $m \times 1$ column vector, the variables of the dual program.

$y^o = (y_1^o, y_2^o, \ldots, y_m^o)'$ , an $m \times 1$ column vector, the optimal solution to dual program.

$Z = v'x$ , a scalar, the value of the primal objective function.

$Z(X)$ , $Z$ is a function of $X$ .

$Z_r = V_r' X_o^r$ , a scalar, the value of the primal objective function when the basis $B_r$ is specified.

$z_j^r = V_r' X_j^r$ , a scalar, the value of the equivalent combination when the basis is $B_r$ .

$\lambda$ a column vector with elements $\lambda_i$ , where $i$ is to be specified wherever it appears.

In addition to the above symbols, we must distinguish the following inequality signs when we are considering set values, e.g. when matrix, vector, or set is involved.

$\geq$ (or $\leq$) means $\geqq$ (or $\leqq$) and $\neq$ , i.e. the strict inequality holds at least once.

Finally, the equation number contains two or three parts. The first part represents the number of chapter, and the second part the order of appearance. The third part, if appears, represents subdivision of equations or program.

# CHAPTER 1

## FORMULATION OF LINEAR PROGRAMS

Let us consider a linear program as

$$\text{Maximize} \quad Z = v'x \tag{1.1.1}$$

$$\text{Subject to} \quad Ax \leqq b \tag{1.1.2}$$

$$x \geqq 0 \tag{1.1.3}$$

where $A = (a_{ij})$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $n > m$, $n$ and $m$ are finite, $x = (x_1,x_2,\ldots,x_n)'$, an $n \times 1$ column vector of variables to be determined, $b = (b_1,b_2,\ldots,b_m)'$ an $m \times 1$ constant column vector, $v = (v_1,v_2,\ldots,v_n)'$ an $n \times 1$ constant column vector. $Z$ is a scalar, value of objective function $v'x$. To avoid confusion we define all vectors as column vectors. If we want row vectors, we need only to transpose our defined vectors.

We can rewrite (1.1.1), (1.1.2), and (1.1.3) as follows

$$\text{maximize} \quad Z = \sum_{j=1}^{n} v_j x_j \tag{1.1.1'}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leqq b_i \tag{1.1.2'}$$

$$i = 1,2,\ldots,m$$

$$x_1,x_2,\ldots,x_n \geqq 0 \tag{1.1.3'}$$

We can partition the coefficient matrix $A$ as [1]

[1] We follow the notation $A_j$ in Dorfman, Samuelson, and Solow, Linear Programming and Economic Analysis, p. 143.

$$A = (A_1, A_2, \ldots A_n)$$

$$\text{where} \quad A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \; j = 1, \ldots, n$$

We call $A_j$ the process vector.

Thus (1.1.2) can be written as [2]

$$\sum_{j=1}^{n} A_j x_j \leqq b$$

i.e. linear combination of $A_j (j = 1,2,\ldots,n)$ , with coefficients $x_j (j = 1,2,\ldots,n)$ . This expression is very important in our later discussion of the simplex method.

For later discussion we formulate the dual problem of the above primal problem as follows

$$\text{Minimize} \quad W = b'y \tag{1.2.1}$$

$$\text{Subject to} \quad A'y \geqq v \tag{1.2.2}$$

$$y \geqq 0$$

where $y = (y_1, y_2, \ldots, y_m)'$ a $m \times 1$ column vector, and

$W$ is a scalar.

$$\text{or Minimize} \quad W = \sum_{i=1}^{m} b_i y_i \tag{1.2.1'}$$

$$\text{Subject to} \quad \sum_{i=1}^{m} a_{ji} y_i \geqq v_j \tag{1.2.2'}$$

---

[2] This kind of expression is very common in the theory of linear programming, e.g. Dorfman, Samuelson, and Solow, op. cit., p. 144.

$$j = 1,2,\ldots,n$$

$$y_1, y_2, \ldots, y_m \geqq 0 \qquad (1.2.3')$$

In words, the dual problem is to find a set of non-negative variables $y_i$ such that $W$, the value of objective function is minimized, under the constraints (1.2.2) and (1.2.3).

The above inequalities (1.1.2') can be transformed into equations by introducing slack variables or surplus variables as follows

$$\begin{array}{l} a_{11}x_1 + \ldots + a_{1n}x_n + x_{n+1} + 0\cdot x_{n+2} + \ldots + 0\cdot x_{n+m} = b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n + 0\cdot x_{n+1} + x_{n+2} + 0\cdot x_{n+3} + \ldots + 0\cdot x_{n+m} = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \quad \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n + 0\cdot x_{n+1} + \ldots + 0\cdot x_{n+m-1} + x_{n+m} = b_m \end{array}$$

where $x_{n+1},\ldots,x_{n+m}$ are the slack variables, and the corresponding process vectors $A_{n+1},\ldots,A_{n+m}$ are unit vectors such that the coefficient matrix becomes

$$A^* = (A,I) \qquad (1.3)$$

and the variable vector and the $v$ vector become

$$x^* = (x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+m})'$$

$$v^* = (v_1,\ldots,v_n,v_{n+1},\ldots,v_{n+m})'$$

$$\text{where } v_{n+1} = v_{n+2} = \ldots = v_{n+m} = 0$$

Thus our primal problem can be written as

$$\text{Max} \qquad v^{*\prime}x^* \tag{1.1.1''}$$

$$\text{Subject to} \quad Z^*x^* = b \tag{1.1.2''}$$

$$x^* \geq 0 \tag{1.1.3''}$$

where (1.1.2") is expressed in equalities. Similarly we can transform the dual problem into the form of equalities, except the new coefficient matrix is

$$(A,-I)$$

We call the added process vectors and the corresponding variables surplus vectors and surplus variables.

The introduction of slack and surplus variables and vectors will help us find the real optimal solution, and will help us undertake table computations. We shall discuss the importance of slack, surplus and artificial variables in Chapter 4.

Finally, notice should be taken of three points:

1. The above formulation is of standard (symmetric) form. If formulated in canonical (unsymmetric) form, the variables of the dual problem need not be constrained to be non-negative.[3]

2. The canonical form is written as

$$\text{max} \qquad Z = v'x \tag{1.3.1}$$

$$\text{subject to} \quad Ax = b \tag{1.3.2}$$

[3] Gale, The Theory of Linear Economic Models, p. 76; or Gass, Linear Programming, p. 83.

$$x \geqq 0 \qquad (1.3.3)$$

This program differs from the primal program (1.1.1), (1.1.2), and (1.1.3) only in (1.3.2), i.e., $Ax = b$ is formulated in equalities, while (1.1.2) in inequalities.

Note (1.1.1"), (1.1.2"), and (1.1.3") can be considered as a canonical form. However, since (1.1.1"), (1.1.2"), and (1.1.3") is transformed from our primal problem (1.1.1), (1.1.2), and (1.1.3), the variables of its dual problem are always constrained to be non-negative. To show this let us write (1.1.1"), (1.1.2"), and (1.1.3") as follows [4]

$$\max Z = v^{*\prime}x^{*} = (v',0') \begin{matrix} x \\ x^{o} \end{matrix} \qquad (1.4.1)$$

$$\text{subject to} \quad (A,I) \begin{matrix} x \\ x^{o} \end{matrix} = b \qquad (1.4.2)$$

$$\begin{matrix} x \\ x^{o} \end{matrix} \geqq 0 \qquad (1.4.3)$$

$$\text{where} \quad 0 = (0_{n+1}, 0_{n+2}, \ldots, 0_{n+m})'$$

$$x^{o} = (x_{n+1}, x_{n+2}, \ldots, x_{n+m})'$$

and the dual program turns out to be

$$\min \quad W = b'y \qquad (1.5.1)$$

$$\text{subject to} \quad \begin{matrix} A' \\ I' \end{matrix} \; y \geqq \begin{matrix} v \\ 0 \end{matrix} \qquad (1.5.2)$$

where $y$ is not constrained to be non-negative

---

[4] Gass, op. cit., pp. 90 - 91. It is worthwhile to point out again the distinction between the transformed canonical form and the original canonical form.

However, if we make a slight manipulation, (1.5.2) can be decomposed into two parts

$$A'y \geqq v \tag{1.5.3}$$

and

$$I'y \geqq 0 \tag{1.5.4}$$

Thus $y$ is necessarily constrained to be non-negative.

Now by the same precedure, the dual problem of (1.3.1) (1.3.2), and (1.3.3) is as follows

$$\min \quad W = b'y$$

$$\text{subject to} \quad A'y \geqq v$$

where $y$ is not constrained to be non-negative.

This time $y$ is really not constrained to be non-negative. The trick is that there is no identity matrix in the coefficient matrix, and there is no null vector in $v$ vector. Consequently, we can conclude that although all the standard form can be transformed into canonical form by introducing slack variables, the transformed canonical form still has the property of the standard form. Therefore we must distinguish the transformed canonical form, (1.1.1"), (1.1.2"), and (1.1.3") from the original canonical form (1.3.1), (1.3.2), and (1.3.3).

3. We shall discuss the maximization problem only, since any minimization problem can be transformed into maximization problem by multiplying the objective function by -1, i.e. multiplying all the $v_i (i = 1,2,\ldots,n)$ by $-1$.

# CHAPTER 2

## THEORY OF LINEAR PROGRAMMING

We have formulated the linear program in Chapter 1. Before we start discussing the simplex method, we discuss the essence of the theory of linear programming briefly in this Chapter. We shall deal with convexity, feasibility, fundamental theorems, duality theorem, existence theorem, and equilibrium theorem.

### §1. Assumptions

First of all we make the following assumptions:

1. Linearity.[1] As formulated in Chapter 1, our system is linear in the sense that it has the properties of additivity and homogeneity. By linearity we mean that a system is subject to the rules of addition and scalar multiplication. For instance, suppose we have a linear function $f(x)$, where $x = x_1 + x_2$, then

$$f(x) = f(x_1 + x_2) = f(x_1) + f(x_2)$$

This is the property of additivity.

Now suppose $\lambda_1$ and $\lambda_2$ are scalars, then

$$f(\lambda_1 x_1 + \lambda_2 x_2) = f(\lambda_1 x_1) + f(\lambda_2 x_2) \quad \text{due to rule of addition}$$

$$= \lambda_1 f(x_1) + \lambda_2 f(x_2) \quad \text{due to rule of scalar multiplication}$$

---

[1] Hadley, Linear Algebra, pp. 1 - 2, and pp. 133 - 134. Note this assumption leads to convexity of our system.

The rule of scalar multiplication has the property of homogeneity. i.e.

$$f(\lambda_1 x_1) = \lambda_1 f(x_1)$$

$$f(\lambda_2 x_2) = \lambda_2 f(x_2)$$

2. Closedness and boundedness. Suppose we have a set

$$T = \{x | x_1^2 + x_2^2 \leqq 1\}$$

then $T$ is closed. $T$ is the entire circle shown below. Any point $x = (x_1, x_2)$ which satisfies $x_1^2 + x_2^2 \leqq 1$ is in $T$. The boundary of the circle is included, therefore it is closed.

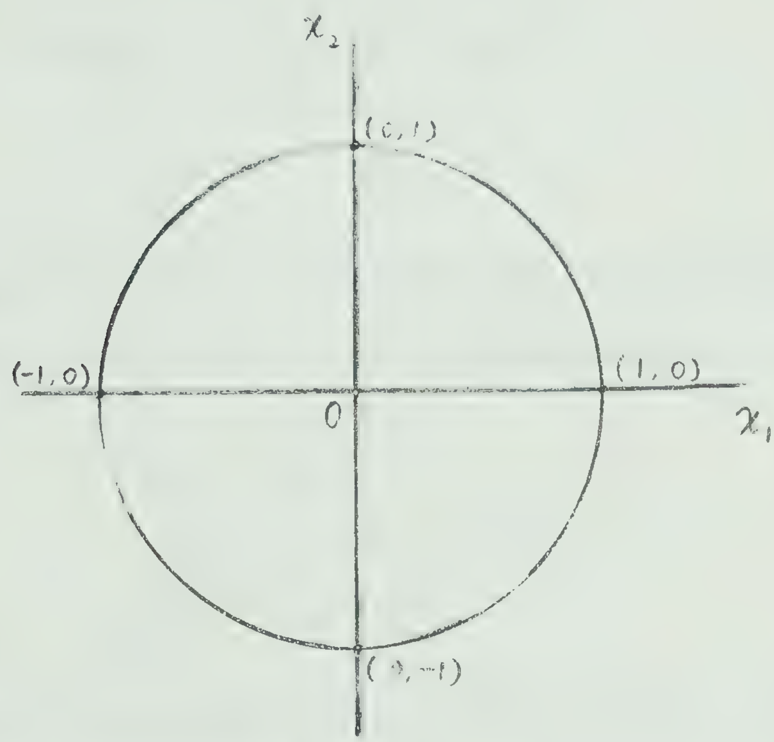



Fig. I

Closedness is very important in maximization (or minimization) problem. If $T$ is not closed, then the boundary of the circle is not included in $T$, i.e. $x_1^2 + x_2^2 < 1$. Suppose a point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ in $T$ is very close to the boundary. If the boundary is not included in $T$, then we can always find a point

$$\bar{x} + \varepsilon = (\bar{x}_1 + \varepsilon_1 , \bar{x}_2 + \varepsilon_2)$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is positive and sufficiently small

such that $\bar{x} + \varepsilon$ is also in $T$. Thus we can never find a point which takes the largest or (smallest) value. Max (or min) position can never be reached. Our linear program is formulated in weak inequalities, so it is closed.

Next, if any point $x$ in $T$ takes finite value, then the system is said to be bounded. We assume our system is bounded. We shall discuss bounded solution in Chapter 4.

3. Consistency. We assume (1.1.2) is consistent, i.e. [2]

$$\rho(A,b) = \rho(A)$$

The augmented matrix $(A,b)$, including the column vector $b$ into $A$, has the same rank as $A$. The following system is not consistent,

$$2x_1 + 2x_2 \leqq 4$$

$$x_1 + x_2 \geqq 3$$

since the rank of coefficient matrix $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ is 1, while the rank of the augmented matrix $\begin{bmatrix} 2 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$ is 2. Graphically as shown below, this system is inconsistent.

[2] We follow the idea of Hadley. Linear Algebra, pp. 167 - 170 and Linear Programming, p. 78.

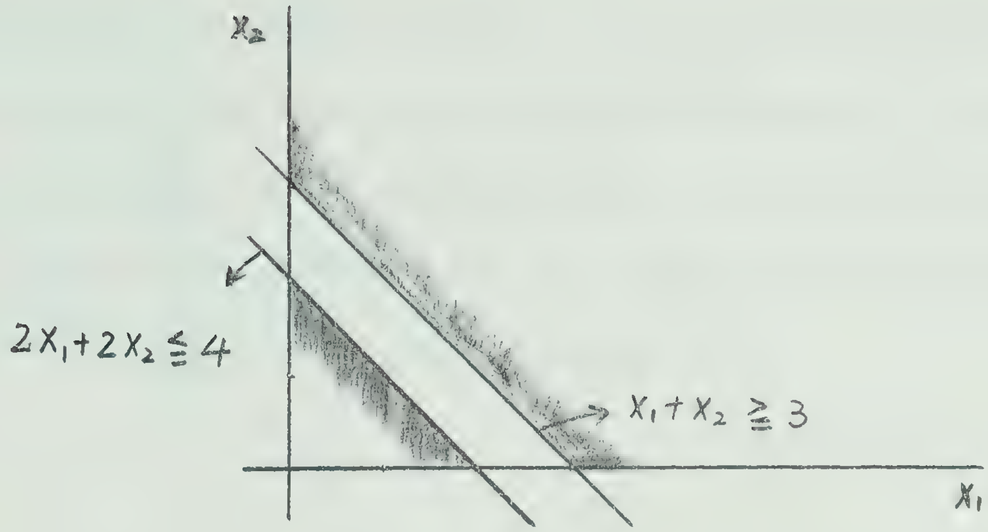



Fig. II

4. We assume linear independence for (1.1.2), i.e.[3]

$$\rho(A) = m \tag{2.3}$$

This condition is made to ensure unique basic solutions. Assumption 4 implies assumption 3, since an m-dimensional vector space can not have more than $m$ linearly independent vectors. This assumption can be relaxed. However, for simplifying our discussion, we shall retain it.

5. Non-degeneracy. i.e. we do not deal with the case where one or more than one zero appears in solution. We must point out that assumption 4 implies this assumption. However, since we shall mention non-degeneracy very frequently, we prefer to make it as an independent assumption.

## §2. Feasibility and convexity

We want to know the properties of the constraints (1.1.2)

---

[3] Hadley, Linear Algebra, pp. 167 - 170.

and (1.1.3) or (1.1.2') and (1.1.3'). When a solution satisfies constraints (1.1.2) and (1.1.3), the solution is said to be feasible. We can consider that all the solutions of our linear program which satisfies (1.1.2) and (1.1.3) construct a feasible set $S$. For illustrating the feasible set $S$, we can consider the following example.

Example 1.

Suppose we are to find $x_1$ and $x_2$ such that

$$Z = x_1 + x_2 \text{ is max} \tag{2.1.1}$$

$$\text{subject to} \quad \begin{aligned} x_1 + 2x_2 &\leq 4 \\ 2x_1 + x_2 &\leq 5 \\ 2x_1 + 0.x_2 &\leq 3 \end{aligned} \tag{2.1.2}$$

$$x_1 \geq 0 , x_2 \geq 0 \tag{2.1.3}$$

We plot (2.1.2) and (2.1.3) on Fig. III, the shaded area OABDE is the feasible set $S$. The point $C$ is not feasible. Any point in $S$ is feasible.

Now the feasible set $S$ is said to be convex if all convex combinations of feasible solutions remain in $S$. In fact $S$ is convex. To show this, suppose there exist two feasible solutions $X_1$ and $X_2$ to (1.1.2). We shall show that a solution which is a convex combination of $X_1$ and $X_2$ is still in $S$. Define

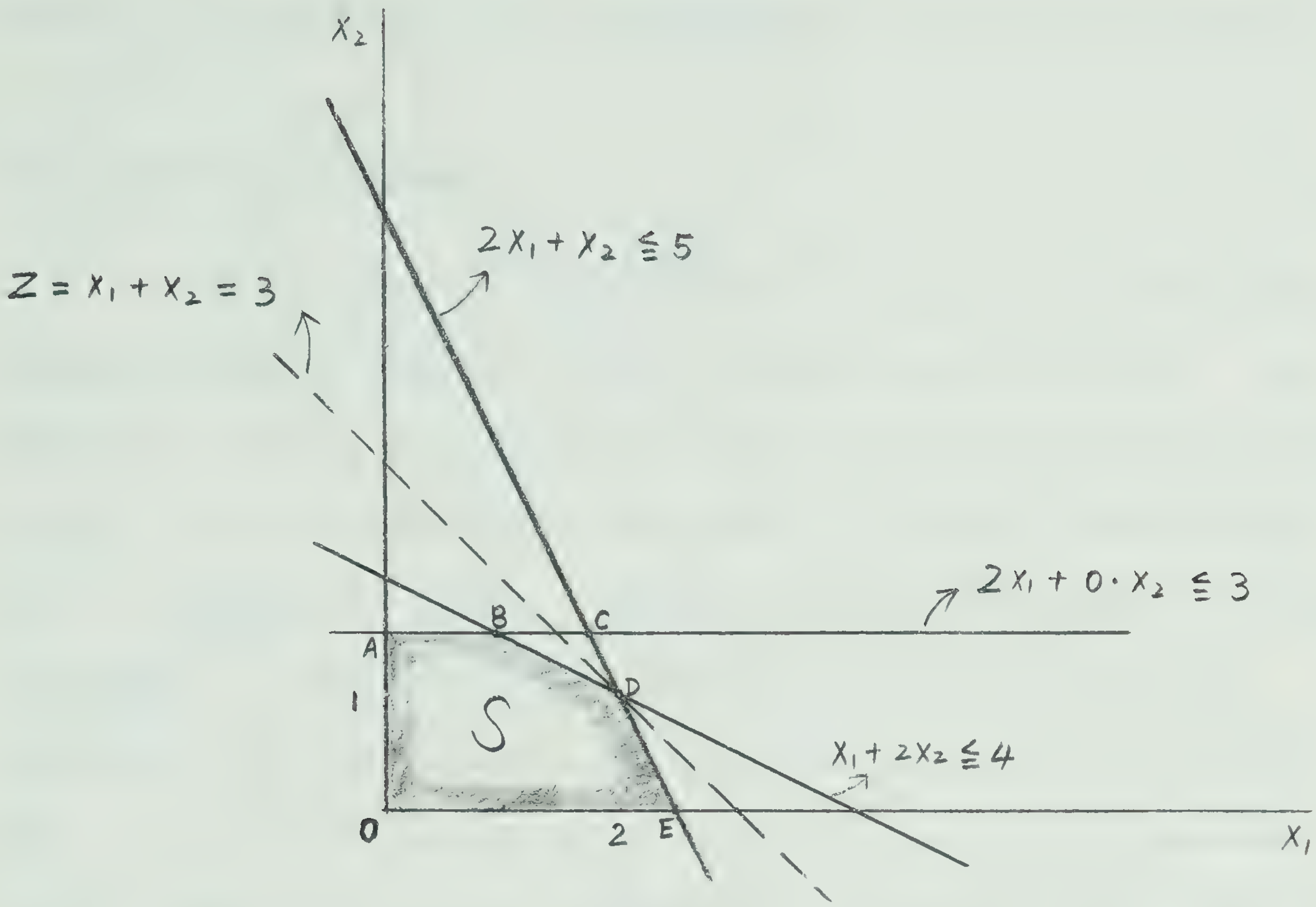



Fig. III

$$x = \lambda_1 x_1 + \lambda_2 x_2$$

$$\text{where} \quad \lambda_1, \lambda_2 \geqq 0 \text{ , and } \lambda_1 + \lambda_2 = 1$$

Since $x_1$ and $x_2$ are two solutions, we have

$$Ax_1 \leqq b \quad \text{and} \quad Ax_2 \leqq b$$

Now $Ax = A(\lambda_1 x_1 + \lambda_2 x_2)$

$$= A[\lambda_1 x_1 + (1 - \lambda_1)x_2] \quad \text{since} \quad \lambda_1 + \lambda_2 = 1$$

$$= \lambda_1 Ax_1 + (1 - \lambda_1)Ax_2 \quad \text{by assumption 1}$$

$$\leqq \lambda_1 b + (1 - \lambda_1)b = b$$

Therefore S is convex.

Now let us look at Fig. III. The shaded area $S$ is convex intuitively.

## §3. Fundamental Theorems

For our algebraic manipulation of $Ax \leqq b$. We replace the inequality with an equality. Since $A$ has dimension of $m \times n$, and $\rho(A) = m$, it is in an m-dimensional vector space. We denote this space by $V_m$. In an m-dimensional vector space $V_m$, there cannot be more than $m$ independent vectors. Suppose the first $m$ columns are linearly independent. Then we can always solve $Ax = b$ for the first $m$ $x_j$ by specifying arbitrary values to the remaining vectors $x_j (j = m+1, m+2, \ldots, n)$, since the remaining vectors $A_j (j = m+1, m+2, \ldots, n)$ can be expressed as linear combination of the $m$ independent vectors which form a basis for $V_m$. As a special case we can set all the remaining $x_j = 0$. This solution is called the basic solution. In fact this is not a special case. We shall show that the remaining $x_j (j = m+1, m+2, \ldots, n)$ must be zero in Theorem 2.1 and Theorem 2.2. Basic solutions cannot have more than $m$ non-zero elements in it. Since we have assumed non-degeneracy, all the basic solutions must have exactly $m$ non-zero elements. From the theory of linear algebra, we know that all the solutions to a linear system are the linear combinations of extreme points, or are the extreme points themselves. Further, all the feasible solutions to our linear system above can be expressed as convex combinations of the extreme points in $S$. On Fig. III, points 0,A,B,D,E are extreme points of $S$. Any feasible solution can be generated by the convex combination of all or some of points 0,A,B,D,E.

Now we are going to show that an extreme point has exactly $m$ non-zero elements in it, and that an extreme point is simply a basic solution to our linear system above. Consequently, we can conclude that all the solutions to our linear system can be generated by basic solutions. Let us prove the following two theorems.

THEOREM 2.1: If a solution $x$ is an extreme point of the feasible set $S$, there can not be more than $m$ non-zero variables in the solution.[4]

Proof: Our proof follows Karlin with our own modification. Suppose there exist $m + 1$ non-zero variables (elements) in a solution, say the first $m + 1$ ones, $x_1 > 0$, $x_2 > 0$, ... $x_m > 0$, $x_{m+1} > 0$. Since the rank of $A$ is $m$, we can always assign an arbitrary value to one of the variables. Thus the solution is not unique. Since $\rho(A) = m$, the first $m + 1$ columns of $A$ are not linearly independent, we can always find a set of scalars $\lambda_j (j = 1,2,...,m+1)$ not all zero, such that

$$\sum_{j=1}^{m+1} \lambda_j A_j = 0 \;, \quad \lambda_j \geqq 0 \quad \text{for} \quad j = 1,2,...,m$$
$$\lambda_{m+1} > 0$$

or in matrix notation

$$\bar{A}\lambda = 0 \tag{2.2}$$

where $\bar{A} = (A_1 A_2 \ldots A_{m+1})$

[4] Karlin, Mathematical Methods and Theory in Games, Programming, and Economics, pp. 161 - 162. Alternative proof can be found in Dorfman, Samuelson, and Solow, op. cit., pp. 75 - 77. In Chapter 3, the idea of finding a new basis can be used as an alternative proof.

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m+1})' \geq 0 \quad \text{with the inequality holding at least once}$$

This is a linear homogeneous system, and $\rho(\overline{A}) = m < m + 1$. Thus we can always find a vector $\lambda$ up to a scalar multiple. Suppose we found two solutions $\lambda^1$ and $\lambda^2$ (vectors)

$$\lambda^1 = \varepsilon\lambda \quad \text{and} \quad \lambda^2 = -\varepsilon\lambda \tag{2.3}$$

where $\varepsilon$ is a constant scalar

Thus (2.2) can be rewritten as

$$\overline{A}(\varepsilon\lambda) = \overline{A}\lambda^1 = 0 \tag{2.4}$$

$$\overline{A}(-\varepsilon\lambda) = \overline{A}\lambda^2 = 0$$

and the corresponding linear system is

$$\overline{A}x = b \tag{2.6}$$

Adding (2.6) to (2.4) and (2.5), we get

$$\overline{A}x + \overline{A}\lambda^1 = b \text{ , or } \overline{A}(x + \lambda^1) = b \tag{2.7}$$

$$\overline{A}x + \overline{A}\lambda^2 = b \text{ , or } \overline{A}(x + \lambda^2) = b \tag{2.8}$$

Thus from (2.7) and (2.8) we get two different solutions $(x + \lambda^1)$ and $(x + \lambda^2)$ with the $m + 1$ st elements being non-zero. However by (2.3)

$$\frac{1}{2}[(x + \lambda^1) + (x + \lambda^2)] = x$$

This result shows that $x$ is a convex combination of $(x + \lambda^1)$ and

$(x + \lambda^2)$, contradicting our assumption that $x$ is an extreme point of the feasible set $S$.

Q.E.D.

Thus we conclude that there can never exist more than $m$ non-zero elements in the extreme point. Now to show there is a one-to-one correspondence between a basic solution and an extreme point, we can prove the following theorem.

THEOREM 2.2: If $x_1 > 0$, $x_2 > 0$, ..., $x_m > 0$, $x_{m+1} = x_{m+2} = \ldots = x_n = 0$, then $x$ is an extreme point in $S$.[5]

Proof: Suppose there exist two different basic solutions $x'$ and $x''$ so that

$$Ax' = b \tag{2.9}$$

$$Ax'' = b \tag{2.10}$$

where $x' = (x'_1, x'_2, \ldots, x'_m, 0_{m+1}, \ldots, 0_n)'$

$x'' = (x''_1, x''_2, \ldots, x''_m, 0_{m+1}, \ldots, 0_n)'$

Subtracting (2.9) from (2.10) we get

$$A(x' - x'') = 0 \tag{2.11}$$

(2.11) is a homogeneous system. If $(x' - x'') \neq 0$, then we can always find a solution, say $x'''$, subject to a linear transformation (by assumption 1)

[5] Karlin, op. cit., p. 162. We follow Karlin more closely than in Theorem 2.1.

$$x''' = x'' + \varepsilon(x' - x'') \tag{1.12}$$

where $\varepsilon$ is an arbitrary scalar

$$x''' = (x_1''', x_2''', \ldots, x_m''', 0_{m+1}, \ldots, 0_n)'$$

Since (2.11) is a linear homogeneous system, $\varepsilon$ is not unique. If $\varepsilon > 0$ and $(x' - x'') < 0$, or if $\varepsilon < 0$ and $(x' - x'') > 0$, then we can increase or decrease $\varepsilon$ arbitrarily until one element of $x'''$, say the mth element $x_m'''$, is equal to zero. Thus one of the first $m$ columns in $A$ becomes dependent on the remaining $m - 1$ columns. This result contradicts our assumption 4, $\rho(A) = m$. Consequently,

$$(x' - x'') = 0$$

i.e. the solution to $Ax = b$ is unique, then $x$ is an extreme point in $S$.

Q.E.D.

Our assumption 4, $\rho(A) = m$, can be relaxed. We can always drop any redundant rows and columns from $A$. However this will lead us to degeneracy case with which we shall not deal.

By Theorem 2.1 and Theorem 2.2, we can conclude that there is a one-to-one correspondence between an extreme point and a basic solution, and that all the solutions can be found in basic solutions, or can be expressed by linear combinations of basic solutions. This conclusion is very important in the theory of linear programming. Since basic solutions are extreme points, any other solution can be generated by linear

combination of the basic solutions. Further since $S$ is convex as shown in section 2, any feasible solution can be generated by convex combination of basic feasible solutions. Thus to solve a linear program, we need to find the basic feasible solutions only.

Next we are going to show that the optimal solution is one of the basic feasible solutions. Let us prove the following theorem.

THEOREM 2.3. If the value of the objective function, $Z_f$ , at an extreme point is optimal, then the value of objective function at any other point in $S$ cannot exceed $Z_f$ .[6]

Proof: Define $X_o^r$ $(r = 1,2,...,k)$ as the extreme points, i.e. basic solutions, in the feasible set $S$ , where

$X_o^r = (x_{10}^r, x_{20}^r, ..., x_{mo}^r)'$ , if the basis $r$ contains the first $m$ process vectors.

Here $r$ is defined as the index number of basis, its maximum possible number can reach

$$k = \binom{n}{m} \tag{2.13}$$

The first subscript of $x_{io}^r$, $i$, is arbitrary, depending on the combination of any $m$ process vectors from $n$ process vectors in $A$ . Now we are to prove Theorem 2.3. Suppose there is a solution $X_o$ which is the convex combination of $X_0^1, X_0^2, ..., X_0^f$, $f \leq k$ , i.e.

---

6 Gass, op. cit., pp. 47 - 49. However our approach of proof is different from that of Gass.

$$X_o = \sum_{r=1}^{f} \lambda_r X_o^r$$

$$\text{where} \quad \lambda_r \geqq 0 \ , \quad \sum_{r=1}^{f} \lambda_r = 1$$

Define $Z(X_o^r)$ the value of objective function as a function of $X_o^r$ . Now suppose $X_o^f$ is the maximizer among $X_o^r(r = 1,2,\ldots,f)$ , then

$$Z(X_o) = Z(\sum_{r=1}^{f} \lambda_r X_o^r) = \sum_{r=1}^{f} \lambda_r Z(X_o^r) \quad \text{by assumption 1 .}$$

$$\leqq \sum_{r=1}^{f} \lambda_r Z(X_o^f) = Z(X_o^f)$$

$$Z(X_o) < Z(X_o^f) \quad \text{when} \quad Z(X_o^r) < Z(X_o^f) \ , \ r \neq f$$

$$\text{or} \quad Z(X_o) = Z(X_o^f) \quad \text{when} \quad Z(X_o^r) = Z(X_o^f) \ , \ r \neq f$$

Q.E.D.

This result shows that the value of the objective function $Z(X_o)$ at a point of convex combination of extreme points in $S$ cannot be larger than the maximum $Z(X_o^r)$ at an extreme point. Analogously for the minimum case. Thus the optimal solution must be $X_o^f$ when $Z(X_o^f) > Z(X_o^r)$ , $r \neq f$ , or some extreme points and their convex combinations when $Z(X_o^f) = Z(X_o^r)$ for some $r$ , $r \neq f$ .

So far we have come to a point where searching for an optimal solution it is necessary only to search from one basic feasible solution to another, until an optimal one is found. Then the procedure is iterative. The maximum possible steps of iteration are

$$k = \binom{n}{m} \quad \text{by} \quad (2.13)$$

since $A$ is an $m \times n$ matrix, and $\rho(A) = m$ , every time we select $m$

independent columns from the $n$ columns in $A$ to form a basis. Further $k$ is finite since $n$ and $m$ are finite as defined in Chapter 1.

Thus we have obtained the results necessary for solving the linear program formulated in Chapter 1. Now return to Example 1 and look at Fig. III. The extreme points are 0,A,B,C,D,E. Here C is not in S , thus basic feasible solutions are 0,A,B,D,E . However it is not interesting to consider point 0 . In addition algebraically points A and E are not the basic solutions in the sense of bases in vector space $V_2$ . Thus we have to find two points only, i.e. B and D . Now the problem is to find a point from B and D such that $Z = x_1 + x_2$ is max. There are only two steps. Obviously, the optimal solution is D , since the highest line $Z = x_1 + x_2$ touches the feasible set S at point D . Graphically the optimal solution is the coordinates of point D , i.e. $x_1 = 2$, $x_2 = 1$ , and the line (or hyperplane in 3 or more dimensional vector space) has a constant term 3, i.e. $Z = 3$ .

## §4 . Existence Theorem, Duality Theorem, Equilibrium Theorem

Before we start discussing the simplex method, we prove the following theorems for subsequent analysis.

EXISTENCE THEOREM: If both the primal and the dual problems have feasible solutions then there exist an optimal solution. [7]

Proof: Recall (1.1.2) and (1.2.2)

$$Ax \leqq b \quad \text{and} \quad A'y \geqq x$$

---

[7] Karlin, op. cit., p. 123. A similar proof can be found in Lancaster, Mathematical Economics, pp. 29 - 30, though the statement of the theorem is slightly different.

Premultiplying (1.1.2) and (1.2.2) by $y'$ and $x'$ respectively, we get

$$y'Ax \leqq y'b \tag{2.14}$$

$$x'A'y \geqq x'v$$

Transposing (2.15) we get

$$y'Ax \geqq v'x \tag{2.15'}$$

By (2.14) and (2.15')

$$v'x \leqq y'Ax \leqq y'b \tag{2.16}$$

$$\text{or} \quad Z(x) \leqq W(x)$$

$$\text{where} \quad Z(x) = v'x \quad \text{as a function of } x \tag{2.17}$$

$$W(y) = y'b \quad \text{as a function of } y \tag{2.18}$$

Suppose $y$ is a feasible solution, then for all feasible solutions $x$, the primal objective function $Z(x)$ is bounded from above. $y'Ax$ is the upper bound by (2.16). Then, since the feasible set is closed and convex, there must exist a maximum. Now if $y$ is not feasible, then for all feasible $x$, $Z(x)$ is not bounded. $Z(x)$ does not have upper bound, since $y'Ax$ does not satisfies the feasibility. Similar argument applies to the dual objective function $W(y)$.

This theorem helps us investigate if our linear program is incorrectly formulated. However, it is difficult to judge the feasibility of both $x$ and $y$. An easy way can be derived from the simplex method.

We shall discuss unbounded solutions in Chapter 3. Next we are to prove the duality theorem.

DUALITY THEOREM: If the primal problem has an optimal solution, then there exists an optimal solution to the dual problem, and the values of both objective functions are equal.[8]

Proof: Denote the basis formed by $m$ columns in $A$ by $B_r$, and the basis which yields the optimal solution by $B_f$. Suppose $B_f$ contains the first $m$ columns in $A$.

$$B_f = (A_1 A_2 \ldots A_m)$$

$$\text{where } f \leqq k$$

The basic equation can be written as

$$B_r X_o^r = b \tag{2.19}$$

and the basic solution is

$$X_o^r = B_r^{-1} b \tag{2.20}$$

Thus the optimal solution can be written as

$$X_o^f = B_f^{-1} b \tag{2.21}$$

and the value of primal objective function is

$$V_f' X_o^f = \max \ Z(\mathrm{x}) \tag{2.22}$$

$$\text{where } V_f = (v_1, v_2, \ldots, v_m)'$$

[8] Gass, op. cit., pp. 84 - 85. We follow Gass only in taking the extreme values of both $Z(\mathrm{x})$ and $W(y)$.

Now suppose we have found the dual basic solution corresponding to the basis $B_f$

$$B_f' y^o = V_f \text{ , where } y^o = (y_1^o, y_2^o, \ldots, y_m^o)' \tag{2.23.1}$$

$$y^o = (B^{-1})' V_f$$

However we do not know if $y^o$ is the optimal solution to the dual problem or not. Premultiplying (2.23.2) by $b'$, and using (2.21) we get

$$b' y^o = b'(B_f^{-1})' V_f = (B_f^{-1} b)' V_f = X_o^{f\prime} V_f \tag{2.24}$$

Transposing (2.24) and using (2.22) we get

$$W(y^o) = y^{o\prime} b = V_f' X_o^f = \max \ Z(x) \tag{2.25}$$

Now recall (2.16'), i.e.

$$Z(x) \leqq W(y)$$

This inequality holds even when we take the extreme values of both sides, i.e.

$$\max Z(x) \leqq \min W(y) \tag{2.26}$$

Now we can consider $Z(x)$ and $W(y)$ as two disjoint sets with only one possible point of intersection if the equality of (2.26) holds. However we do not know whether there exists such an intersection or not. We still do not know if $W(y^o)(y^{o\prime}b)$ is equal to min $W(y)$ or not. However both $W(y^o)$ and $\min W(y)$ are in the set $W(y)$. Now by (2.25) and (2.26), we have

$$W(y^o) = \max Z(x) \leqq \min W(y) \tag{2.27}$$

However by definition we know

$$W(y^o) \geqq \min W(y) \tag{2.28}$$

By (2.27) and (2.28), the two inequalities hold simultaneously, thus

$$W(y^o) = \min W(y)$$

$$\text{i.e.} \quad y^{o'}b = \min W(y)$$

$$\max Z(x) = \min W(y)$$

$$\text{since} \quad y^{o'}b = \max Z(x) \quad \text{by (2.25)}$$

Q.E.D.

Thus the duality theorem can be used as a criterion for judging the optimality. We shall return to the duality theorem in Chapter 4.

Note that the notations used in the above proof will be used throughout our discussion of the simplex method in Chapters 3 and 4. Special attention should be paid to the index number of basis.

Finally before we start discussing the simplex method, we prove the following theorem for later analysis.

EQUILIBRIUM THEOREM: If $\overline{x}$ is the optimal solution to the primal problem, and $\overline{y}$ is the optimal solution to its dual problem, then [9]

---

[9] Karlin, op. cit., p. 124. Similar proof can be found in any book on the theory of linear programming, e.g. Gale, op. cit., pp. 82 - 83.

$$\sum_{j=1}^{n} a_{ij}\bar{x}_j < b_i \quad \text{implies} \quad \bar{y}_i = 0$$

for any $i(= 1,2,\ldots,m)$

and $\sum_{i=1}^{m} a_{ji}\bar{y}_i > v_j$ implies $\bar{x}_j = 0$

for any $j(= 1,2,\ldots,n)$

where $\bar{x} = (\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_m,0_{m+1},\ldots,0_n)'$

$\bar{y} = (\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_m)'$

Proof: Premultiplying (2.19) by $y^{o\prime}$, we get

$$y^{o\prime}B_fX_o^f = y^{o\prime}b \ , \ r = f \tag{2.29}$$

Premultiplying (2.23.1) by $X_o^{f\prime}$, we get

$$X_o^{f\prime}B_f'y^o = X_o^{f\prime}V_f \tag{2.30}$$

Transposing (2.30)

$$y^{o\prime}B_fX_o^f = V_f'X_o^f \tag{2.30'}$$

By (2.29) and (2.30') we have

$$V_f'X_o^f = y^{o\prime}B_fX_o^f = y^{o\prime}b \tag{2.31}$$

Since we have proved that $X_o^f$ and $y^o$ are the optimal solutions to the primal and dual problems respectively, the equality (2.31) represents optimality.

Now note that $\bar{x}$ is simply $X_o^f$, except the difference in dimension, i.e.

$$\bar{x} = \begin{bmatrix} X_o^f \\ 0 \end{bmatrix} \quad \text{an} \quad n \times 1 \quad \text{column vector}$$

where $0 = (0_{m+1}, 0_{m+2}, \ldots, 0_n)'$

$\bar{y}$ is $m \times 1$ column vector, exactly the same as $y^o$ in Duality Theorem. However in order to be consistent with $\bar{x}$, we use $\bar{y}$ instead of $y^o$.

However we must note that the dimension does not matter in optimality. (2.31) can be written as

$$V'\bar{x} = \bar{y}'A\bar{x} = \bar{y}'b \tag{2.32}$$

or in scalar form

$$V'\bar{X} = v_1\bar{x}_1 + v_2\bar{x}_2 + \ldots + v_m\bar{x}_m + v_{m+1}\cdot 0 + \ldots + v_n\cdot 0$$

$$= \sum_{j=1}^{n} v_j\bar{x}_j \left(= \sum_{j=1}^{m} v_j\bar{x}_j\right)$$

dimension does not matter.

$$\bar{y}'b = \bar{y}_1 b_1 + \bar{y}_2 b_2 + \ldots + \bar{y}_m b_m = \sum_{i=1}^{m} \bar{y}_i b_i$$

and $\bar{y}'A\bar{x} = \sum_{i=1}^{m} \bar{y}_i \left( \sum_{j=1}^{n} a_{ij}\bar{x}_j \right)$

$$= \sum_{i=1}^{m} \bar{y}_i (a_{i1}\bar{x}_1 + a_{i2}\bar{x}_2 + \ldots + a_{im}\bar{x}_m + a_{im+1}\cdot 0 + \ldots + a_{in}\cdot 0)$$

$$= \bar{x}_1 \sum_{i=1}^{m} a_{i1}\bar{y}_i + \bar{x}_2 \sum_{i=1}^{m} a_{i2}\bar{y}_i + \ldots + \bar{x}_m \sum_{i=1}^{m} a_{im}\bar{y}_i$$

$$+ 0 \cdot \sum_{i=1}^{m} a_{im+1}\bar{y}_i + \ldots + 0 \cdot \sum_{i=1}^{m} a_{in}\bar{y}_i$$

$$= \sum_{j=1}^{n} \bar{x}_j \left( \sum_{i=1}^{m} a_{ij}\bar{y}_i \right) , \text{ or } = \sum_{j=1}^{m} \bar{x}_j \left( \sum_{i=1}^{m} a_{ij}\bar{y}_i \right)$$

Again dimension does not matter.

Thus (2.32) can be written as

$$\sum_{j=1}^{n} v_j\overline{x}_j (= \sum_{j=1}^{m} v_j\overline{x}_j) = \sum_{j=1}^{n} \overline{x}_j(\sum_{i=1}^{m} a_{ij}\overline{y}_i)\{= \sum_{j=1}^{m} \overline{x}_j(\sum_{i=1}^{m} a_{ij}\overline{y}_i)\}$$

$$= \sum_{i=1}^{m} \overline{y}_i b_i \tag{2.32'}$$

(2.32') justifies that (2.32) is identical with (2.31). The reason why we use (2.32) instead of (2.31) is that we want to explain the equilibrium of the whole system, not only of a basis, though the final basis $B_f$ represents optimality, i.e. equilibrium.

Now by manipulating the first two terms of (2.32) we get

$$v'\overline{x} - \overline{y}'A\overline{x} = 0 \ , \ (v' - \overline{y}'A)\overline{x} = 0 \tag{2.33.1}$$

and manipulating the last two terms of (2.32) we get

$$\overline{y}'A\overline{x} - \overline{y}'b = 0 \ , \ \overline{y}'(A\overline{x} - b) = 0 \tag{2.33.2}$$

(2.33.1) and (2.33.2) can be written respectively as

$$\sum_{j=1}^{n} (v_j - \sum_{i=1}^{m} a_{ji}\overline{y}_i)\overline{x}_j = 0 \tag{2.33.1'}$$

$$\sum_{i=1}^{m} \overline{y}_i(\sum_{j=1}^{n} a_{ij}x_j - b_i) = 0 \tag{2.33.2'}$$

Now from (2.33.1')

$$\text{if} \quad v_j - \sum_{i=1}^{m} a_{ji}\overline{y}_i < 0 \ , \ \text{then} \quad \overline{x}_j = 0$$

for any $j(= 1,2,...,n)$

$$\text{if} \quad \sum_{j=1}^{n} a_{ij}\bar{x}_j - b_i < 0 \text{ , then } \bar{y}_i = 0$$

$$\text{for any } i(= 1,2,\ldots,m)$$

Q.E.D.

This result is very useful in economic theory. We shall interpret this result in Chapter 6.

# CHAPTER 3

## THEORY OF THE SIMPLEX METHOD

Now the next step is to solve our linear program. From Theorems 2.1, 2.2, 2.3, we know that if there exists an optimal solution, it must be one of the finite number of basic feasible solutions. We can try all the possible number of bases, i.e. $k$ bases, and then find the optimal one among $k$. This is called the Complete Description Method. For our later discussion, we denote it by Method A. However by trying Method A, the computational task is tremendously tedious, e.g. if A is a 7 x 3 matrix, then

$$k = \binom{7}{3} = 35 \quad \text{by (2.13)}$$

i.e. we have to try 35 basic solutions. To get rid of this difficulty, the Simplex Method is commonly used.

The procedure of the simplex method is as follows: Starting with a basis, choose one process vector not in the initial basis to replace a process vector in the basis at each step of iteration which continues until the value of the objective function reaches the optimum subject to an optimality criterion to be discussed later in this Chapter.

In the simplex method, we always deal with the transformed canonical form. i.e. (1.1.1")(1.1.2") and (1.1.3"). Let us discuss the essence of the theory of the simplex method in this chapter, and the computation technique in Chapter 4.

## §1. Initial Basic Feasible Solution and Its Equivalent Combinations

Suppose we start with the first $m$ columns in $A^*$, where

$$A^* = (A_1, A_2, \ldots A_n, A_{n+1}, \ldots, A_{n+m}) = (A, I) \quad \text{by (1.3)}$$

Then the initial basis is

$$B_r = (A_1 A_2 \ldots A_m) \ , \ r = 0$$

The corresponding basic equations turn out to be

$$B_r X_o^r = b \ , \ r = 0 \tag{3.1}$$

Thus the initial basic feasible solution is

$$X_o^r = X_o^o = B_o^{-1} b > 0 \tag{3.2}$$

where $X_o^o = (x_{10}^o, x_{20}^o, \ldots, x_{mo}^o)'$, an $m \times 1$ column vector, which is restricted to be positive, since we have assumed non-degeneracy in Chapter 2.

(3.1) can be written as

$$\sum_{i=1}^{m} x_{io}^o A_i = b \tag{3.1'}$$

vector $b$ is expressed as a positive (it must be non-negative in general case) linear combination of the first $m$ independent column vectors in $A^*$, i.e. $b$ is in the convex cone formed by the independent vectors $A_i (i = 1,2,\ldots m)$ in m-space, $V_m$.

Since $A_i (i = 1,2,\ldots,m)$ are independent vectors in $V_m$, any vector in $V_m$ other than $A_i$ $(i = 1,2,\ldots,m)$ can be expressed as linear combination of $A_i (i = 1,2,\ldots,m)$. Thus

$$\sum_{i=1}^{m} x_{ij}^{o} A_i = A_j \tag{3.3}$$

where $j = m + 1$, $m + 2$, ..., $m + n$, and $x_{ij}^{o}$ $(i = 1,2,\ldots,m)$ are, not necessarily non-neg, called the equivalent combination to one unit of $A_j$.

Suppose we now select arbitrarily a vector $A_t$, $t > m$, to replace a vector, $A_s$, $s \leqq m$. Then, by multiplying (3.3) by an arbitrary positive constant $\theta$, which is to be determined later in this section, and moving the right hand side to the left hand side, we get

$$\sum_{i=1}^{m} \theta x_{it}^{o} A_i - \theta A_t = 0 \tag{3.4}$$

Subtracting (3.4) from (3.1'), we get

$$\sum_{i=1}^{m} (x_{io}^{o} - \theta x_{it}^{o}) + \theta A_t = b \tag{3.5}$$

Here $A_t$ and $A_i (i = 1,2,\ldots,m)$ are to form a new basis. We have to find a solution such that it remains in the feasible set $S$. To do so $\theta$ must be non-negative, since if it is negative the solutions corresponding to $A_t$, $x_{to}^{r}$, is negative, i.e. $\theta = x_{to}^{r}$, $r = 1$. If $\theta$ is zero, the new basis is simply the old one. Therefore we restrict $\theta$ to be strictly positive. If all $x_{it}^{o} \leq 0$ $(i = 1,2,\ldots m)$ with the strict inequality holding at least once, then the new solution is feasible, since $x_{io}^{o}$ is

positive by (3.2) . If at least one $x^o_{it}$ is positive, it seems to be able to choose a $\theta$ sufficiently small, such that the new solution still remains in the feasible set $S$ .

Now since $A_i (i = 1,2,\ldots,m)$ are independent vectors in $V_m$ , $A_t$ is a linear combination of $A_i$ , $A_t$ , $A_1$ , $A_2$ , ... , $A_m$ are not linearly independent. Therefore the solution to (3.5), $\theta$ and $(x^o_{io} - \theta x^o_{it})$ $(i = 1,2,\ldots,m)$ are not unique. However we know $x^o_{io}$ and $x^o_{it}$ are uniquely determined by (3.1') and (3.3). Consequently, $\theta$ can be any positive number. Suppose there is at least one $x^o_{it} > 0$ , and $\theta$ is very small. If we increase gradually the value of $\theta$ , then we can end up with one of $(x^o_{io} - \theta x^o_{it})$ equal to zero. The vector corresponding to the zero $(x^o_{io} - \theta x^o_{it})$ , say $(x^o_{so} - \theta x^o_{st})$ , is then the vector to be removed from the old basis $B_o$ . Then the solution is unique, since the $m - 1$ independent vectors and any other vector in $V_m$ can always form a basis for $V_m$ .

If all the $x^o_{it} \leq 0$ $(i = 1,2,\ldots,m)$ with the strict inequality holding at least once, then we can choose a vector such as $A_t$ to replace a vector such as $A_s$ . However, in this case, the solution is unbounded,[1] since $\theta$ can be increased arbitrarily without coming to a point where one of $(x^o_{io} - \theta x^o_{it})$ becomes zero.

Therefore boundedness is as important as closedness, convexity, etc. in finding optimal solution to linear program. Our system is formulated in weak inequalities, therefore it is automatically closed. Our assumption of linearity in Chapter 2 leads us to convexity. However

---

[1] Hadley, Linear Programming, pp. 93 - 95. Our discussion of boundedness follows Hadley in some points.

without any further assumption, our system is not necessarily bounded. We have assumed boundedness in Chapter 2. However we want to know what condition leads us to boundedness assumption. We shall be back to boundedness in the next section.

## §2. New Basic Feasible Solution, and Optimality Criterion

Now, how do we determine $\theta$ ? To show this explicitly, let us rewrite (3.3) as

$$x^o_{1t}A_1 + x^o_{2t}A_2 + \ldots + x^o_{st}A_s + \ldots + x^o_{mt}A_m = A_t$$

Dividing by $x^o_{st}$ and removing all terms except $A_s$ to the right hand side, we get

$$A_s = \frac{1}{x^o_{st}}A_t - \frac{1}{x^o_{st}}\sum_{\substack{i=1\\ i\neq s}}^{m} x^o_{it}A_i$$

$$\text{if} \quad x^o_{st} \neq 0$$

Substituting in (3.1') and in (3.3), we get the following respectively,

$$(x^o_{10} - \frac{x^o_{so}}{x^o_{st}} x_{1t})A_1 + (x^o_{20} - \frac{x^o_{so}}{x^o_{st}} x^o_{2t})A_2 + \ldots + (x^o_{s-1\cdot 0} - \frac{x^o_{so}}{x^o_{st}} x^o_{s-1\cdot t})A_{s-1}$$

$$+ (x^o_{s+1\cdot 0} - \frac{x^o_{so}}{x^o_{st}} x^o_{s+1\cdot t})A_{s+1} + \ldots + (x^o_{mo} - \frac{x^o_{so}}{x^o_{st}} x^o_{mt})A_m + \frac{x^o_{so}}{x^o_{st}} A_t = b$$

or $$\sum_{i=1}^{m} (x^o_{io} - \frac{x^o_{so}}{x^o_{st}} x^o_{it})A_i + \frac{x^o_{so}}{x^o_{st}} A_t = b \qquad (3.6.1)$$

where the sth term $(x^o_{so} - \frac{x^o_{so}}{x^o_{st}} x^o_{st})A_s = 0$

and

$$(x^o_{1j} - \frac{x^o_{sj}}{x^o_{st}} x^o_{1t})A_1 + (x^o_{2j} - \frac{x^o_{sj}}{x^o_{st}} x^o_{2t})A_2 + \ldots + (x^o_{s-1\cdot j} - \frac{x^o_{sj}}{x^o_{st}} x^o_{s-1\cdot j})A_{s-1}$$

$$+ (x^o_{s+1\cdot j} - \frac{x^o_{sj}}{x^o_{st}} x^o_{s+1\cdot t})A_{s+1} + \ldots + (x^o_{mj} - \frac{x^o_{sj}}{x^o_{st}} x^o_{mt})A_m + \frac{x^o_{sj}}{x^o_{st}} A_t = A_j$$

or $$\sum_{i=1}^{m} (x^o_{ij} - \frac{x^o_{sj}}{x^o_{st}} x^o_{it})A_i + \frac{x^o_{sj}}{x^o_{st}} A_t = A_j \tag{3.6.2}$$

where the sth term $(x^o_{sj} - \frac{x^o_{sj}}{x^o_{st}} x^o_{st})A_s = 0$

By comparing (3.6.1) with (3.5)

$$\theta = \frac{x^o_{so}}{x^o_{st}} \tag{3.7.1}$$

$$\text{if} \quad x^o_{st} \neq 0 \tag{3.7.2}$$

To ensure the solution $\frac{x^o_{so}}{x^o_{st}}$ and $(x^o_{io} - \frac{x^o_{so}}{x^o_{st}} x_{it})$ for $i \neq s$ to be in the feasible set $S$, $\frac{x^o_{so}}{x^o_{st}}$ must be strictly positive as mentioned before. Since we have assumed non-degeneracy, $x^o_{so}$ must be positive. Thus $x^o_{st}$ must also be positive. Consequently in order to have condition (3.7.2) satisfied, the following strict inequality must hold.

$$x^o_{st} > 0 \tag{3.8}$$

This is simply the condition for boundedness.

Now to ensure the solution, $\frac{x^o_{so}}{x^o_{st}}$ and $(x^o_{io} - \frac{x^o_{so}}{x^o_{st}} x^o_{it})$ to be feasible, we can choose $\frac{x^o_{so}}{x^o_{st}}$ such that it is the smallest among $\frac{x^o_{io}}{x^o_{it}}$

for all $i$ corresponding to strictly positive $x^o_{it}$ , i.e.

$$\theta = \frac{x^o_{so}}{x^o_{st}} = \min_i \left(\frac{x^o_{io}}{x^o_{it}} , \text{ for all positive } x^o_{it}\right) \tag{3.9}$$

Thus we can conclude that if at least one $x^o_{it}$ is positive among all $i$ , then it is possible to obtain a new basic feasible solution. Further we can also conclude that if at least one $x^r_{ij}$ is positive for all $i$ in the old basis, and all $j$ not in the old basis, for all bases involved, then there exists an optimal solution. Consequently we can restate our boundedness assumption as follows:

$$x^r_{ij} > 0 \tag{3.10.1}$$

for all $i$ , depending on $r$

and for all $j \neq i$ , at each $r(\leqq k)$

Contrarily, as mentioned above, if

$$x^r_{ij} \leq 0 \tag{3.10.2}$$

for all $i = 1,2,...,m$, and all $j > m$ ,

with the strict inequality holding at least once

then we can neither find a new basic feasible solution, nor an optimal solution.

We must note that (3.5) does not necessarily imply that a new basis can be found, since $\theta$ can be any scalar, it may not necessarily be able to be determined by rule (3.9). However (3.6.1) and (3.8) imply

that there exists a new basis which satisfies the feasibility condition. Since the $m$ vectors $A_t$ and $A_i (i \neq s)$ are linearly independent the solution is unique and bounded. Further we also must note that $\theta$ may not be unique according to the rule (3.9), there may be more than one $x_{io}^o / x_{it}^o$ have the same minimum value. If this situation occurs, there is at least one zero component in the solution, i.e. degeneracy. Since we have assumed non-degeneracy, $\theta$ must be unique.

Determination of $\theta$ determines what variable in (3.5) is to be zero. In other words it determines what process vector in the old basis is to be removed.

So far we have come to a position where we have started with an initial basis to replace a vector in the basis according to the rule (3.9). Now we should like to know how to choose a vector outside the basis, i.e. $A_j$ , at each step of iteration, and what an optimal position looks like.

Recall (3.1') and (3.5). The objective function of (3.1') is

$$Z_r = \sum_{i=1}^{m} v_i x_{io}^r , \quad r = 0 \tag{3.11}$$

Now when we introduce a new vector $A_t$ not in the basis, the new solution is $\theta$ and $(x_{io}^o - \theta x_{it}^o)$ . $i = 1,2,\ldots,m$ . The new value of objective function turns out to be

$$Z_r = \sum_{i=1}^{m} v_i (x_{io}^o - \theta x_{it}^o) + v_t \theta \tag{3.12}$$

where we define $r = 1$ for the new basis index

number, and remember the smallest $(x_{io}^o - \theta x_{it}^o)$, $i = 1,2,\ldots,m$, is zero.

By slight manipulation, (3.12) can be rewritten as

$$Z_1 = \sum_{i=1}^{m} v_i x_{io}^o + v_t \theta - \theta \sum_{i=1}^{m} v_i x_{it}^o$$

$$= Z_o + \theta(v_t - \sum_{i=1}^{m} v_i x_{it}^o) \qquad (3.12')$$

since the first term is nothing but $Z_o$ by (3.11)

By moving $Z_o$ to the left hand side,

$$Z_1 - Z_o = \theta(v_t - \sum_{i=1}^{m} v_i x_{it}^o) \qquad (3.13)$$

which simply means that the increase in the value of $Z_r$ is $\theta(v_t - \sum_{i=1} v_i x_{it}^o)$ when a new vector $A_t$ is introduced.

Now recall (3.3). $x_{it}^o (i = 1,2,\ldots,m)$ are simply combinationally equivalent to one unit of $A_t$. Consequently, $\sum_{i=1}^{m} v_i x_{it}^o$ can be interpreted as the value of objective function reduced (or increased if $\sum_{i=1}^{m} v_i x_{it}^o < 0$) by per unit new vector $A_t$, while $v_t$ can be interpreted as the value of objective function contributed by one unit of $A_t$.

Define $$z_t^r = \sum_{i=1}^{m} v_i x_{it}^r \qquad (3.14)$$

Then, since $\theta > 0$, if $v_t - z_t^r > 0$, it means $Z_r$ has an increase in the value of objective function. If $v_t - z_t^r \leqq 0$, then there is no increase of $Z_r$. In general, as long as

$$v_j - z_j^r > 0 \quad \text{for some} \quad j > m \tag{3.15}$$

our procedure of iteration will continue. Finally, if

$$v_j - z_j^r \leqq 0 \quad \text{for all} \quad j > m \tag{3.16}$$

Then the optimal position is reached. Let us call (3.16) the optimality criterion.

Thus to select a vector outside the old basis to be introduced into the new basis is simply to select the vector corresponding to the largest positive $(v_j - z_j^r)$ , $j > m$ , since its contribution to $Z_r$ is the greatest. Consequently, $v_t - z_t^r$ is the largest among all $(v_j - z_j^r)$ , $j > m$ .

# CHAPTER 4

## EXAMPLES AND TABLE COMPUTATION

In this chapter we shall use two examples for illustrating the theory of the simplex method discussed above. In addition we shall discuss two techniques of table computation and an alternative optimality criterion.

### §1. Examples

Example 2.[1]

$$\max Z = 8x_1 + 19x_2 + 7x_3 \tag{4.1.1}$$

$$\text{subject to} \quad \begin{aligned} 3x_1 + 4x_2 + x_3 &\leqq 10 \\ x_1 + 3x_2 + 3x_3 &\leqq 20 \end{aligned} \tag{4.1.2}$$

$$x_1, x_2, x_3 \geqq 0 \tag{4.1.3}$$

As mentioned in Chapter 1, since (4.1.2) is in inequalities we can transform it into equalities by introducing slack variables. The introduction of slack variables may help us find optimal solution when there is no basis, formed by non-slack vectors, which yields optimal solution. Let us transform (4.1.2) into equations by introducing slack variables $x_4$ and $x_5$ , then the above linear program becomes

$$\max Z = 8x_1 + 19x_2 + 7x_3 + 0 \cdot x_4 + 0 \cdot x_5 \tag{4.1.1'}$$

---

[1] Example 2 is an exercise in Gale, op. cit, p. 130, EXERCISE 6. We have changed the original vector b = (25,50)' to b = (10,20)', for the convenience of our subsequent geometric analysis in Chapter 3, since it is difficult to draw the smaller vectors such as (1,0)',(0,1)',(1.3)', and (4.3)' compared with b = (25,50)' on a graph paper.

$$\text{subject to} \quad 3x_1 + 4x_2 + x_3 + x_4 + 0 \cdot x_5 \qquad (4.1.2')$$

$$x_1 + 3x_2 + 3x_3 + 0 \cdot x_4 + x_5$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \qquad (4.1.3')$$

There are five process vectors

$$A_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \; A_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \; A_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; A_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let us try Method A first. The maximum possible number of bases is

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10 \quad \text{by (2.13)}$$

We can find all the basic feasible solutions, and then select an optimal solution among them. Define the ten possible bases as follows:

$$\begin{aligned} B_1 &= (A_1A_2), \; B_2 = (A_1A_3), \; B_3 = (A_1A_4) \\ B_4 &= (A_1A_5), \; B_5 = (A_2A_3), \; B_6 = (A_2A_4) \qquad (4.2) \\ B_7 &= (A_2A_5), \; B_8 = (A_3A_4), \; B_9 = (A_3A_5) \\ B_{10} &= (A_4A_5) \end{aligned}$$

Now let us start with $B_1 = (A_1A_2)$ to obtain basic solutions. Apply (3.1)

$$B_1X_o^1 = b$$

$$\text{where} \quad B_1 = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \quad \Delta_1 = |B_1| = 5$$

$$B_1^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ -1 & 3 \end{bmatrix} \qquad (4.3.1)$$

$$X_0^1 = \begin{bmatrix} x_{10}^1 \\ x_{20}^1 \end{bmatrix} = B_1^{-1}b = \begin{bmatrix} -10 \\ 10 \end{bmatrix} \qquad (4.3.2)$$

which is not feasible.

Next try $B_2 = (A_1 A_3)$

$$B_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \Delta_2 = 8$$

$$B_2^{-1} = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \qquad (4.3.3)$$

$$X_0^2 = \begin{bmatrix} x_{10}^2 \\ x_{30}^2 \end{bmatrix} = B_2^{-1}b = \begin{bmatrix} 1\frac{1}{4} \\ 6\frac{1}{4} \end{bmatrix} > 0 \qquad (4.3.4)$$

which is feasible.

The value of objective function turns out to be

$$Z_2 = v_1 x_{10}^2 + v_3 x_{30}^2 = 53\frac{3}{4} \qquad (4.3.5)$$

Try $B_3 = (A_1 A_4)$

$$B_3 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \quad \Delta_3 = -1$$

$$B_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \qquad (4.3.6)$$

$$X_0^3 = \begin{bmatrix} x_{10}^3 \\ x_{40}^3 \end{bmatrix} = \begin{bmatrix} 20 \\ -50 \end{bmatrix} \qquad (4.3.7)$$

which is not feasible.

Try $B_4 = (A_1 A_5)$

$$B_4 = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad , \quad \Delta_4 = 3$$

$$B_4^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} \tag{4.3.8}$$

$$X_0^4 = \begin{bmatrix} x_{10}^4 \\ x_{50}^4 \end{bmatrix} = \begin{bmatrix} 3\frac{1}{3} \\ 16\frac{2}{3} \end{bmatrix} > 0 \tag{4.3.9}$$

which is feasible.

$$Z_4 = v_1 x_{10}^4 + v_5 x_{50}^4 = 26\frac{2}{3} \tag{4.3.10}$$

Try $B_5 = (A_2 A_3)$

$$B_5 = \begin{bmatrix} 4 & 1 \\ 3 & 3 \end{bmatrix} \quad , \quad \Delta_5 = 9$$

$$B_5^{-1} = \frac{1}{9}\begin{bmatrix} 3 & -1 \\ -3 & 4 \end{bmatrix} \tag{4.3.11}$$

$$X_0^5 = \begin{bmatrix} x_{20}^5 \\ x_{30}^5 \end{bmatrix} = \begin{bmatrix} 1\frac{1}{9} \\ 5\frac{5}{9} \end{bmatrix} > 0 \tag{4.3.12}$$

which is feasible.

$$Z_5 = v_2 x_{20}^5 + v_3 x_{30}^5 = 60 \tag{4.3.13}$$

Try $B_6 = (A_2 A_4)$

$$B_6 = \begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix} \quad , \quad \Delta_6 = -3$$

$$B_6^{-1} = -\frac{1}{3}\begin{bmatrix} 0 & -1 \\ -3 & 4 \end{bmatrix} \tag{4.3.14}$$

$$X_0^6 = \begin{bmatrix} x_{20}^6 \\ x_{40}^6 \end{bmatrix} = \begin{bmatrix} 20/3 \\ -50/3 \end{bmatrix} \tag{4.3.15}$$

which is not feasible.

Try $B_7 = (A_2 A_5)$

$$B_7 = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}, \quad \Delta_7 = 4$$

$$B_7^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix} \tag{4.3.16}$$

$$X_0^7 = \begin{bmatrix} x_{20}^7 \\ x_{50}^7 \end{bmatrix} = \begin{bmatrix} 2\frac{1}{2} \\ 12\frac{1}{2} \end{bmatrix} > 0 \tag{4.3.17}$$

which is feasible.

$$Z_7 = v_2 x_{20}^7 + v_5 x_{50}^7 = 47\frac{1}{2} \tag{4.3.18}$$

Try $B_8 = (A_3 A_4)$

$$B_8 = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}, \quad \Delta_8 = -3$$

$$B_8^{-1} = -\frac{1}{3}\begin{bmatrix} 0 & -1 \\ -3 & 1 \end{bmatrix} \tag{4.3.19}$$

$$X_0^8 = \begin{bmatrix} x_{30}^8 \\ x_{40}^8 \end{bmatrix} = \begin{bmatrix} 6\frac{2}{3} \\ 3\frac{1}{3} \end{bmatrix} > 0 \tag{4.3.20}$$

which is feasible

$$Z_8 = v_3 x_{30}^8 + v_4 x_{40}^8 = 46\tfrac{2}{3} \tag{4.3.21}$$

Try $B_9 = (A_3 A_5)$

$$B_9 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad \Delta_9 = 1$$

$$B_9^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \tag{4.3.22}$$

$$X_0^9 = \begin{bmatrix} x_{30}^9 \\ x_{50}^9 \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \end{bmatrix} \tag{4.3.23}$$

which is not feasible

Finally, since in $B_{10} = (A_4 A_5)$, both $A_4$ and $A_5$ are slack vectors, it is meaningless to choose a basis whose component vectors are all slack.

So far we have obtained five basic feasible solutions and their values of objective functions as follows:

$$X_0^2 = \begin{bmatrix} x_{10}^2 \\ x_{30}^2 \end{bmatrix} = \begin{bmatrix} 1\frac{1}{4} \\ 6\frac{1}{4} \end{bmatrix}, \quad Z_2 = 53\tfrac{3}{4}$$

$$X_0^4 = \begin{bmatrix} x_{10}^4 \\ x_{50}^4 \end{bmatrix} = \begin{bmatrix} 3\frac{1}{3} \\ 16\frac{2}{3} \end{bmatrix}, \quad Z_4 = 26\tfrac{2}{3}$$

$$X_0^5 = \begin{bmatrix} x_{20}^5 \\ x_{30}^5 \end{bmatrix} = \begin{bmatrix} 1\frac{1}{9} \\ 5\frac{5}{9} \end{bmatrix}, \quad Z_5 = 60 \tag{4.3.24}$$

$$X_0^7 = \begin{bmatrix} x_{20}^7 \\ x_{50}^7 \end{bmatrix} = \begin{bmatrix} 2\frac{1}{2} \\ 12\frac{1}{2} \end{bmatrix}, \quad Z_7 = 47\tfrac{1}{2}$$

$$X_0^8 = \begin{bmatrix} x_{30}^8 \\ x_{40}^8 \end{bmatrix} = \begin{bmatrix} 6\frac{2}{3} \\ 3\frac{1}{3} \end{bmatrix}, \quad Z_8 = 46\frac{2}{3}$$

Thus the maximum $Z$ is $Z_5 = 60$ with $A_2$ and $A_3$ in the final basis.

However, the above approach by Method A is tedious. Especially when the number of constraints and the number of variables are large.

Let us try the alternative method, the Simplex Method. Suppose we start with $B_2 = (A_1 A_3)$ , and apply (3.1')

$$A_1 x_{10}^2 + A_3 x_{30}^2 = b \tag{4.4}$$

where to avoid confusion we use index number $r$ consistent with that defined in (4.2). However it need not be consistent. $r$ is used to indicate the step of iteration in simplex method. In table computation to be discussed in Chapter 4 we shall use the index number $r$ as the order of step of iteration.

The initial basic feasible solution to (4.4), and the value of objective function turns out to be

$$X_0^2 = \begin{bmatrix} x_{10}^2 \\ x_{30}^2 \end{bmatrix} = \begin{bmatrix} 1\frac{1}{4} \\ 6\frac{1}{4} \end{bmatrix} \tag{4.5.1}$$

$$Z_2 = 53\frac{3}{4} \tag{4.5.2}$$

Next apply (3.3) for all $A_j$ not in the initial basis $B_2$ , i.e. for $A_2$ , $A_4$ , and $A_5$ ,

$$x_{12}^2 A_1 + x_{32}^2 A_3 = A_2$$

$$x_{14}^2 A_1 + x_{34}^2 A_3 = A_4 \tag{4.6}$$

$$x_{15}^2 A_1 + x_{35}^2 A_3 = A_5$$

Then solve (4.6) for $X_2^2$, $X_4^2$, $X_5^2$ respectively. The inverse matrix of $B_2$ is, by using (4.3.3)

$$B_2^{-1} = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$X_2^2 = \begin{bmatrix} x_{12}^2 \\ x_{32}^2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9/8 \\ 5/8 \end{bmatrix}$$

$$X_4^2 = \begin{bmatrix} x_{14}^2 \\ x_{34}^2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/8 \\ -1/8 \end{bmatrix}$$

$$X_5^2 = \begin{bmatrix} x_{15}^2 \\ x_{35}^2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/8 \\ 3/8 \end{bmatrix}$$

Now apply (3.15) for selecting a vector to be introduced into new basis,

$$\begin{aligned} z_2^2 &= v_1 x_{12}^2 + v_3 x_{32}^2 = 107/8 \\ v_2 - z_2^2 &= 19 - 107/8 = 45/8 > 0 \end{aligned} \tag{4.7.1}$$

$$\begin{aligned} z_4^2 &= v_1 x_{14}^2 + v_3 x_{34}^2 = 17/8 \\ v_4 - z_4^2 &= 0 - 17/8 = -17/8 < 0 \end{aligned} \tag{4.7.2}$$

$$\begin{aligned} z_5^2 &= v_1 x_{15}^2 + v_3 x_{35}^2 = 13/8 \\ v_5 - z_5^2 &= 0 - 13/8 = -13/8 < 0 \end{aligned} \tag{4.7.3}$$

The only $(v_j - z_j^2)$ that satisfies (3.15) is $v_2 - z_2^2$. Consequently, $A_2$ is to be introduced into new basis. To find a vector to be removed from the old basis $B_2$ we apply (3.9)

$$\theta = \min\left(\frac{x_{10}^2}{x_{12}^2}, \frac{x_{30}^2}{x_{32}^2}\right) = \min\left(\frac{25}{9}, 25\right)$$

$$= \frac{x_{10}^2}{x_{12}^2} \qquad (4.8)$$

since the positive $x_{i2}^2$ are $x_{12}^2$ and $x_{32}^2$.

$A_1$ is to be removed from $B_2$ by $A_2$.

The new basis turns out to be

$$B_5 = (A_2 A_3)$$

The inverse matrix of $B_1$ is

$$B_5^{-1} = \frac{1}{9}\begin{bmatrix} 3 & -1 \\ -3 & 4 \end{bmatrix} \text{ from (4.3.11)}$$

The new basic feasible solution is

$$X_0^5 = \begin{bmatrix} x_{20}^5 \\ x_{30}^5 \end{bmatrix} = B_5^{-1} b = \begin{bmatrix} 1\frac{1}{9} \\ 5\frac{5}{9} \end{bmatrix} > 0 \text{ by using (4.3.12)}$$

Apply (3.3) again for all $A_j$ not in $B_5$, i.e. for $A_1$, $A_4$, and $A_5$

$$x_{21}^5 A_2 + x_{31}^5 A_3 = A_1$$

$$x_{24}^5 A_2 + x_{34}^5 A_3 = A_4 \qquad (4.9)$$

$$x_{25}^5 A_2 + x_{35}^5 A_3 = A_5$$

Solve (2,33) for, the equivalent combinations to $A_1$, $A_4$, and $A_5$, $X_1^5$, $X_4^5$, and $X_5^5$ respectively.

$$X_1^5 = \begin{bmatrix} x_{21}^5 \\ x_{31}^5 \end{bmatrix} = B_5^{-1}A_1 = \frac{1}{9}\begin{bmatrix} 3 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8/9 \\ -5/9 \end{bmatrix}$$

$$X_4^5 = \begin{bmatrix} x_{24}^5 \\ x_{34}^5 \end{bmatrix} = B_5^{-1}A_4 = \frac{1}{9}\begin{bmatrix} 3 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}$$

$$X_5^5 = \begin{bmatrix} x_{25}^5 \\ x_{35}^5 \end{bmatrix} = B_5^{-1}A_5 = \frac{1}{9}\begin{bmatrix} 3 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 4/9 \end{bmatrix}$$

Apply (3.15) again

$$\begin{aligned} z_1^5 &= v_2x_{21}^5 + v_3x_{31}^5 = 117/9 \\ v_1 - z_1^5 &= 8 - 117/9 = -5 < 0 \end{aligned} \quad (4,10,1)$$

$$\begin{aligned} z_4^5 &= v_2x_{24}^5 + v_3x_{34}^5 = 4 \\ v_4 - z_4^5 &= -4 < 0 \end{aligned} \quad (4,10,2)$$

$$\begin{aligned} z_5^5 &= v_2x_{25}^5 + v_3x_{35}^5 = 1 \\ v_5 - z_5^5 &= -1 < 0 \end{aligned} \quad (4,10,3)$$

All the $(v_j - z_j^5)$ satisfy the optimality criterion (3.16). Therefore the optimal solution is

$$X_0^5 = \begin{bmatrix} x_{20}^5 \\ x_{30}^5 \end{bmatrix} = \begin{bmatrix} 1\frac{1}{9} \\ 5\frac{5}{9} \end{bmatrix} \quad (4.11.1)$$

$$Z_5 = 60 \tag{4.11.2}$$

The final basis contains $A_2$ and $A_3$ . The results are exactly identical with those obtained by Method A.

Thus as shown above, by using the Simplex Method, we have found the optimal solution in only two steps of iteration, while by using Method A, nine steps (ten including the last slack basis) are needed in Example 2. The Simplex Method requires more computation per iteration, however, the most painstaking task is to compute inverse matrices of bases. At each step of iteration, there is one inverse matrix to be computed. Especially when the number of constraints and the number of variables are large, reduction of steps of iteration is needed desperately. We shall show some efficient computation methods for Simplex Method in next two sections.

Since we are trying to reduce the number of steps of iteration, we may wonder why we introduce the slack variables such that the possible number of bases is increased. In Example 2, the optimal solution does not contain slack variables. However, we may frequently find some slack variables appearing in the optimal solution. The trouble is that we do not know whether it is the case before solving the problem. For illustrating the importance of introducing slack variables, let us solve the following linear program.

Example 3. [2]

---

[2] Example 3 is an exercise in Hadley, Linear Programming, p. 145, Exercise 4 - 14.

$$\text{maximize} \quad Z = 3x_1 + 4x_2 + x_3 + 7x_4 \tag{4.12.1}$$

$$\text{subject to} \quad 8x_1 + 3x_2 + 4x_3 + x_4 \leqq 7$$

$$2x_1 + 6x_2 + x_3 + 5x_4 \leqq 3 \tag{4.12.2}$$

$$x_1 + 4x_2 + 5x_3 + 2x_4 \leqq 8$$

$$x_1, x_2, x_3, x_4 \geqq 0 \tag{4.12.3}$$

First we transform inequalities (4.12.2) into equations by introducing slack variables $x_5$, $x_6$, and $x_7$, then this linear program becomes

$$\text{Maximize} \quad Z = 3x_1 + 4x_2 + x_3 + 7x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 0 \cdot x_7 \tag{4.12.1'}$$

$$\text{Subject to} \quad 8x_1 + 3x_2 + 4x_3 + x_4 + x_5 + 0 \cdot x_6 + 0 \cdot x_7 = 7$$

$$2x_1 + 6x_2 + x_3 + 5x_4 + 0 \cdot x_5 + x_6 + 0 \cdot x_7 = 3 \tag{4.12.2'}$$

$$x_1 + 4x_2 + 5x_3 + 2x_4 + 0 \cdot x_5 + 0 \cdot x_6 + x_7 = 8$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geqq 0 \tag{4.12.3'}$$

If we solve this linear program by Method A, we have to try 34 bases (35 including the basis with all process vectors being slack, i.e. $\binom{7}{3} = 35$). We shall solve this linear program by the more efficient Simplex Method to get rid of this difficulty. Suppose we start with the first three process vectors as the initial basis. Define the initial basis as

$$B_1 = (A_1 A_2 A_3) = \begin{bmatrix} 8 & 3 & 4 \\ 2 & 6 & 1 \\ 1 & 4 & 5 \end{bmatrix} \tag{4.13.1}$$

To obtain the initial basic solution, we apply (3.1)

$$B_1 X_0^1 = b$$

where $X_0^1 = (x_{10}^1, x_{20}^1, x_{30}^1)'$

Compute the inverse matrix of $B_1$

$$\Delta_1 = |B_1| = 189$$

$$B_1^{-1} = \begin{bmatrix} 26 & 1 & -21 \\ -9 & 36 & 0 \\ 2 & -29 & 42 \end{bmatrix} \tag{4.13.2}$$

$$X_0^1 = \begin{bmatrix} x_{10}^1 \\ x_{20}^1 \\ x_{30}^1 \end{bmatrix} = B_1^{-1} b = \begin{bmatrix} 17/189 \\ 45/189 \\ 263/189 \end{bmatrix} > 0 \tag{4.13.3}$$

This is the initial basic feasible solution.

To obtain the equivalent combinations to all $A_j$ , $j > m$ , i.e. to $A_4$ , $A_5$ , $A_6$ , and $A_7$ , we apply (3.3)

$$B_1 X_4^1 = A_4$$

$$B_1 X_5^1 = A_5$$

$$B_1 X_6^1 = A_6$$

$$B_1 X_7^1 = A_7$$

$$\text{or} \quad x^1_{14}A_1 + x^1_{24}A_2 + x^1_{34}A_3 = A_4$$

$$x^1_{15}A_1 + x^1_{25}A_2 + x^1_{35}A_3 = A_5 \tag{4.14}$$

$$x^1_{16}A_1 + x^1_{26}A_2 + x^1_{36}A_3 = A_6$$

$$x^1_{17}A_1 + x^1_{27}A_2 + x^1_{17}A_3 = A_7$$

Solve (4.14) for $X^1_4$ , $X^1_5$ , $X^1_6$ , and $X^1_7$ by using $B^{-1}_1$ from (4.13.2)

$$X^1_4 = \begin{bmatrix} x^1_{14} \\ x^1_{24} \\ x^1_{34} \end{bmatrix} = B^{-1}_1 A_4 = \begin{bmatrix} -11/189 \\ 171/189 \\ -59/189 \end{bmatrix}$$

$$X^1_5 = \begin{bmatrix} x^1_{15} \\ x^1_{25} \\ x^1_{35} \end{bmatrix} = B^{-1}_1 A_5 = \begin{bmatrix} 26/189 \\ -9/189 \\ 2/189 \end{bmatrix}$$

$$X^1_6 = \begin{bmatrix} x^1_{16} \\ x^1_{26} \\ x^1_{36} \end{bmatrix} = B^{-1}_1 A_6 = \begin{bmatrix} 1/189 \\ 36/189 \\ -29/189 \end{bmatrix}$$

$$X^1_7 = \begin{bmatrix} x^1_{17} \\ x^1_{27} \\ x^1_{37} \end{bmatrix} = B^{-1}_1 A_7 = \begin{bmatrix} -21/189 \\ 0 \\ 42/189 \end{bmatrix}$$

Now apply (3.15) to determine which process vector to be introduced into new basis.

$$z_4^1 = v_1 x_{14}^1 + v_2 x_{24}^1 + v_3 x_{34}^1 = 592/189$$

$$v_4 - z_4^1 = 7 - 592/189 = 731/189 > 0$$

$$z_5^1 = v_1 x_{15}^1 + v_2 x_{25}^1 + v_3 x_{35}^1 = 44/189$$

$$v_5 - z_5^1 = 0 - 44/189 = -44/189 < 0$$

$$z_6^1 = v_1 x_{16}^1 + v_2 x_{26}^1 + v_3 x_{36}^1 = 118/189$$

$$v_6 - z_6^1 = 0 - 118/189 = -118/189 < 0$$

$$z_7^1 = v_1 x_{17}^1 + v_2 x_{27}^1 + v_3 x_{37}^1 = -21/189$$

$$v_7 - z_7^1 = 0 - (-21/189) = 21/189 > 0$$

The largest positive $v_j - z_j^1$ is $v_4 - z_4^1$, therefore $A_4$ is to be introduced into new basis. Next let us determine which process vector to be removed from $B_1$. Apply (3.9)

$$\theta = \min_i \left( \frac{x_{i0}^1}{x_{i4}^1} \quad \text{for all} \quad x_{i4}^1 > 0 \right)$$

$$= \frac{x_{20}^1}{x_{24}^1}$$

since the only positive $x_{i4}^1$ is $x_{24}^1$.

Thus the vector in the initial basis to be replaced by $A_4$ is $A_2$. The new basis then turns out to be

$$B_2 = (A_1 A_4 A_3) = \begin{bmatrix} 8 & 1 & 4 \\ 2 & 5 & 1 \\ 1 & 2 & 5 \end{bmatrix} \qquad (4.15.1)$$

$$\Delta_2 = |B_2| = 171$$

$$B_2^{-1} = \frac{1}{171}\begin{bmatrix} 23 & 3 & -20 \\ -9 & 36 & \\ -1 & -15 & 38 \end{bmatrix} \tag{4.15.2}$$

Then the new basic solution is

$$X_0^2 = \begin{bmatrix} x_{10}^2 \\ x_{40}^2 \\ x_{30}^2 \end{bmatrix} = B_2^{-1}b = \begin{bmatrix} 10/171 \\ 45/171 \\ 252/171 \end{bmatrix} > 0 \tag{4.15.3}$$

To obtain the equivalent combinations to all $A_j$ not in $B_2$, i.e. to $A_2$, $A_5$, $A_6$, and $A_7$, we apply (3.3) again

$$\begin{aligned} x_{12}^2A_1 + x_{42}^2A_4 + x_{32}^2A_3 &= A_2 \\ x_{15}^2A_1 + x_{45}^2A_4 + x_{35}^2A_3 &= A_5 \\ x_{16}^2A_1 + x_{46}^2A_4 + x_{36}^2A_3 &= A_6 \\ x_{17}^2A_1 + x_{47}^2A_4 + x_{37}^2A_3 &= A_7 \end{aligned} \tag{4.16}$$

Solve (4.16) for $X_2^2$, $X_5^2$, $X_6^2$, and $X_7^2$ respectively by using the inverse matrix of $B_2$ from (4.15.2)

$$X_2^2 = \begin{bmatrix} x_{12}^2 \\ x_{42}^2 \\ x_{32}^2 \end{bmatrix} = B_2^{-1}A_2 = \begin{bmatrix} 7/171 \\ 189/171 \\ 59/171 \end{bmatrix}$$

$$X_5^2 = \begin{bmatrix} x_{15}^2 \\ x_{45}^2 \\ x_{35}^2 \end{bmatrix} = B_2^{-1}A_5 = \begin{bmatrix} 23/171 \\ -9/171 \\ -1/171 \end{bmatrix}$$

$$X_6^2 = \begin{bmatrix} x_{16}^2 \\ x_{46}^2 \\ x_{36}^2 \end{bmatrix} = B_2^{-1}A_6 = \begin{bmatrix} 3/171 \\ 36/171 \\ -15/171 \end{bmatrix}$$

$$X_7^2 = \begin{bmatrix} x_{17}^2 \\ x_{47}^2 \\ x_{37}^2 \end{bmatrix} = B_2^{-1}A_7 = \begin{bmatrix} -20/171 \\ 0 \\ 38/171 \end{bmatrix}$$

Apply (3.15) again to determine which process vector to be introduced into the next new basis.

$$z_2^2 = v_1x_{12}^2 + v_4x_{42}^2 + v_3x_{32}^2 = 1403/171$$

$$v_2 - z_2^2 = 4 - 1403/171 = -719/171 < 0$$

$$z_5^2 = v_1x_{15}^2 + v_4x_{45}^2 + v_3x_{35}^2 = 5/171$$

$$v_5 - z_5^2 = 0 - 5/171 = -5/171 < 0$$

$$z_6^2 = v_1x_{16}^2 + v_4x_{46}^2 + v_3x_{36}^2 = 246/171$$

$$v_6 - z_6^2 = 0 - 246/171 = -246/171 < 0$$

$$z_7^2 = v_1x_{17}^2 + v_4x_{47}^2 + v_3x_{37}^2 = -22/171$$

$$v_7 = z_7^2 = 0 - (-22/171) = 22/171 > 0$$

The only $v_j - z_j^2$ which satisfies (3.15) is $v_7 - z_7^2$, therefore the vector not in $B_2$ to be introduced into the next new basis is $A_7$. We must note that $A_7$ is a slack process vector. Next

we are to determine which vector is to be removed from $B_2$. Apply (3.9) again.

$$\theta = \min\left(\frac{x_{20}^2}{x_{i7}^2} \quad \text{for all} \quad x_{i7}^2 > 0\right)$$

$$= \frac{x_{30}^2}{x_{37}^2} \quad \text{since the only positive } x_{i7}^2 \text{ is } x_{37}^2 .$$

Consequently, the vector to be replaced with $A_7$ is $A_3$. The new basis then turns out to be

$$B_3 = (A_1 A_4 A_7) = \begin{bmatrix} 8 & 1 & 0 \\ 2 & 5 & 0 \\ 1 & 2 & 1 \end{bmatrix} \tag{4.17.1}$$

$$\Delta_3 = |B_3| = 38$$

$$B_3^{-1} = \frac{1}{38}\begin{bmatrix} 5 & -1 & 0 \\ -2 & 8 & 0 \\ -1 & -15 & 38 \end{bmatrix} \tag{4.17.2}$$

To obtain the equivalent combinations to all $A_j$ not in $B_3$, i.e. to $A_2$, $A_3$, $A_5$, and $A_6$, we apply (3.3) again.

$$\begin{aligned} x_{12}^3 A_1 + x_{42}^3 A_4 + x_{72}^3 &= A_2 \\ x_{13}^3 A_1 + x_{43}^3 A_4 + x_{73}^3 A_7 &= A_3 \\ x_{15}^3 A_1 + x_{45}^3 A_4 + x_{75}^3 A_7 &= A_5 \\ x_{16}^3 A_1 + x_{46}^3 A_4 + x_{76}^3 A_7 &= A_6 \end{aligned} \tag{4.18}$$

Solve (4.18) for $X_2^3$, $X_3^3$, $X_5^3$, and $X_6^3$ respectively by using the inverse matrix of $B_3$ from (4.17.2)

$$X_2^3 = \begin{bmatrix} x_{12}^3 \\ x_{42}^3 \\ x_{72}^3 \end{bmatrix} = B_3^{-1}A_2 = \begin{bmatrix} 9/38 \\ 42/38 \\ 59/38 \end{bmatrix}$$

$$X_3^3 = \begin{bmatrix} x_{13}^3 \\ x_{43}^3 \\ x_{73}^3 \end{bmatrix} = B_3^{-1}A_3 = \begin{bmatrix} 19/38 \\ 0 \\ 171/38 \end{bmatrix}$$

$$X_5^3 = \begin{bmatrix} x_{15}^3 \\ x_{45}^3 \\ x_{75}^3 \end{bmatrix} = B_3^{-1}A_5 = \begin{bmatrix} 5/38 \\ -2/38 \\ -1/38 \end{bmatrix}$$

$$X_6^3 = \begin{bmatrix} x_{16}^3 \\ x_{46}^3 \\ x_{76}^3 \end{bmatrix} = B_3^{-1}A_6 = \begin{bmatrix} -1/38 \\ 8/38 \\ -15/38 \end{bmatrix}$$

Apply (3.15) again

$$z_2^3 = v_1x_{12}^3 + v_4x_{42}^3 + v_7x_{72}^3 = 321/38$$

$$v_2 - z_2^3 = 4 - 321/38 = -169/38 < 0$$

$$z_3^3 = v_1x_{13}^3 + v_4x_{43}^3 + v_7x_{73}^3 = 57/38$$

$$v_3 - z_3^3 = 1 - 57/38 = -19/38 < 0$$

$$z_5^3 = v_1x_{15}^3 + v_4x_{45}^3 + v_7x_{75}^3 = 1/38$$

$$v_5 - z_5^3 = 0 - 1/38 = -1/38 < 0$$

$$z_6^3 = v_1 x_{16}^3 + v_4 x_{46}^3 + v_7 x_{76}^3 = 53/38$$

$$v_6 - z_6^3 = 0 - 53/38 = -53/38 < 0$$

All the $v_j - z_j^3$ are negative. Consequently, by (3.16) we have reached the optimal position. The optimal solution is

$$X_0^3 = \begin{bmatrix} x_{10}^3 \\ x_{40}^3 \\ x_{70}^3 \end{bmatrix} = B_3^{-1} b = \begin{bmatrix} 32/38 \\ 10/38 \\ 252/38 \end{bmatrix} > 0 \qquad (4.19.1)$$

which is certainly feasible.

The value of objective function turns out to be

$$Z = 4 \frac{7}{19} \qquad (4.19.2)$$

The final basis contains $A_1$, $A_4$, and $A_7$ .

So far we have tried only three steps of iteration by Simplex Method, rather than 35 steps by Method A.

We must note here that the final basis contains one slack process vector, $A_7$ . This result indicates that all the other bases containing non-slack process vectors are less favorable than $B_3$ . In other words, by employing only two process vectors, leaving one process vector unused, it is still more attractive than full-employment of three process vectors. Therefore if we did not introduce the slack vectors, and just tried all the possible bases by choosing three vectors from $A_1$, $A_2$, $A_3$, and $A_4$ at each time, we would have ended up with an optimal solution whose $Z$ is less than that of the above optimal solution, even

though no slack process vector is employed. To verify the above argument, we can try Method A, without introducing slack variables $x_5$, $x_6$. amd $x_7$. The maximum possible number of bases is $\binom{4}{3} = 4$.

$$\begin{aligned} B_1 &= (A_1 A_2 A_3) \\ B_2 &= (A_1 A_2 A_4) \\ B_3 &= (A_1 A_3 A_4) \\ B_4 &= (A_2 A_3 A_4) \end{aligned} \tag{4.20}$$

where the subscripts of $B_r$ need not be consistent with those of $B_r$ in (4.13.1) and (4.15.1)

The basic feasible solutions with respect to $B_1$ and $B_3$ are

$$X_0^1 = \begin{bmatrix} x_{10}^1 \\ x_{20}^1 \\ x_{30}^1 \end{bmatrix} = \begin{bmatrix} 17/189 \\ 45/189 \\ 263/189 \end{bmatrix} > 0 \quad \text{by using (4.13.3)}$$

$$X_0^3 = \begin{bmatrix} x_{10}^3 \\ x_{30}^3 \\ x_{40}^3 \end{bmatrix} = \begin{bmatrix} 10/189 \\ 252/171 \\ 45/171 \end{bmatrix} > 0 \quad \text{by using (4.15.3)}$$

To obtain the basic solutions involving $B_2$ and $B_4$, we apply (3.1) to solve the following basic equations

$$B_2 X_0^2 = b \tag{4.21.1}$$

where

$$B_2 = \begin{bmatrix} 8 & 3 & 1 \\ 2 & 6 & 5 \\ 1 & 4 & 2 \end{bmatrix} , \quad X_0^2 = (x_{10}^2, x_{20}^2, x_{40}^2)'$$

and $B_4 X_0^4 = b$ where (4.21.2)

$$B_4 = \begin{bmatrix} 3 & 4 & 1 \\ 6 & 1 & 5 \\ 4 & 5 & 2 \end{bmatrix} , \quad X_0^4 = (x_{20}^4, x_{30}^4, x_{40}^4)'$$

To solve (4.21.1) and (4.21.2) for $X_0^2$ and $X_0^4$ respectively. We first compute the inverse matrices of $B_2$ and $B_4$ .

$$\Delta_2 = |B_2| = -59$$

$$\Delta_4 = |B_4| = -11$$

$$B_2^{-1} = \frac{-1}{59} \begin{bmatrix} -8 & -2 & 9 \\ 1 & 15 & -38 \\ 2 & -29 & 42 \end{bmatrix} \tag{4.21.3}$$

$$B_4^{-1} = \frac{-1}{11} \begin{bmatrix} -23 & -3 & 19 \\ 8 & 2 & -9 \\ 26 & 2 & -21 \end{bmatrix} \tag{4.21.4}$$

Now by using (4.21.3) and (4.21.4) the basic solutions are

$$X_0^2 = \begin{bmatrix} x_{10}^2 \\ x_{20}^2 \\ x_{40}^2 \end{bmatrix} = B_2^{-1} b = \begin{bmatrix} -10/59 \\ 252/59 \\ -263/59 \end{bmatrix}$$

$$X_0^4 = \begin{bmatrix} x_{20}^4 \\ x_{30}^4 \\ x_{40}^4 \end{bmatrix} = B_4^{-1} b = \begin{bmatrix} 18/11 \\ 10/11 \\ -17/11 \end{bmatrix}$$

Both $X_0^2$ and $X_0^4$ are not feasible.

Consequently the values of objective functions for $X_0^1$ and $X_0^3$ are

$$Z_1 = v_1 x_{10}^1 + v_2 x_{20}^1 + v_3 x_{30}^1 = 2\,\frac{116}{189}$$

$$Z_3 = v_1 x_{10}^3 + v_3 x_{30}^3 + v_4 x_{40}^3 = 3\,\frac{84}{171}$$

$$Z_3 > Z_1$$

Recall (4.19.2), Z of the true optimal solution of Example 3 is

$$Z = 4\,\frac{7}{19} > 3\,\frac{84}{171} = Z_3$$

Thus without introducing the slack variables, we can not find the true optimal solution for Example 3.

We shall deal with the slack variables further in next two sections where table computation for simplex method is demonstrated. The computations for the simplex method demonstrated so far is still tedious and laborious, especially when the number of constraints and the number of variables are large. Therefore a more efficient computation method for simplex method is needed.

## §2. Basis Formed by Slack Vectors as Initial Basis

We have mentioned the importance of slack variables in Chapters 1 and 2, and in the preceding section in this chapter. Now we are back to the discussion of slack variables again. Recall Example 2. Suppose we start with an initial basis as

$$B_o = (A_4 A_5) = I$$

i.e. $B_o$ is an identity matrix

Then our initial basic equations become

$$B_o X_o^o = b \tag{4.22}$$

$$\text{or} \quad I X_o^o = b$$

We shall use the index number of basis $r$ by the order of step of iteration. The initial basic solution turns out to be

$$X_o^o = b \tag{4.23}$$

Now apply (3.3) for all $A_j (j = 1,2,\ldots,5)$ to obtain equivalent combinations

$$\begin{aligned} B_o X_1^o &= A_1 \\ B_o X_2^o &= A_2 \\ B_o X_3^o &= A_3 \\ B_o X_4^o &= A_4 \\ B_o X_5^o &= A_5 \end{aligned} \tag{4.24}$$

(Theoretically we need those equivalent combinations to $A_j$ not in the basis only).

$$\begin{aligned} \text{or} \quad I X_1^o &= A_1 \\ I X_2^o &= A_2 \\ I X_3^o &= A_3 \end{aligned} \tag{4.24'}$$

$$IX_4^0 = A_4$$

$$IX_5^0 = A_5$$

Then

$$X_1^0 = A_1 \; , \; X_2^0 = A_2 \; , \; X_3^0 = A_3 \; , \; X_4^0 = A_4 \; , \; X_5^0 = A_5 \tag{4.25}$$

Interestingly from (4.23) and (4.25) the basic solution $X_o^o$ and the equivalent combinations $X_1^0$, $X_2^0$, $X_3^0$, $X_4^0$, $X_5^0$ with respect to $B_o$ are simply the column vectors b, $A_1$, $A_2$, $A_3$, $A_4$, and $A_5$ respectively. Thus if we start with an identity matrix as the initial basis, and at each step of iteration transform each new basis into an identity matrix, then all the basic solutions and equivalent combinations needed can be found from the appropriate column vectors as in (4.23) and (4.25). We can make a table as Table I to show this procedure.

On Table I at the first step (i.e. r = 0), the columns headed by b and $A_i (i = 1,2,...,5)$ represent the basic solution and the equivalent combinations as shown in (4.23) and (4.25). To obtain the initial basic equations (4.22) we can take the inner product of the column headed by B and the column headed by b as follows

$$10A_4 + 20A_5 = b$$

where the column headed by B represents basis, and the basic solution

$$X_o^o = \begin{bmatrix} x_{40}^0 \\ x_{50}^0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

| r | v | B | $A_1$ | $A_2$ | $A_3$ | b | $A_4$ | $A_5$ |
|---|---|---|---|---|---|---|---|---|
| 0 | $v_4 = 0$ | $A_4$ | 3 | 4 | 1 | 10 | 1 | 0 |
| | $v_5 = 0$ | $A_5$ | 1 | 3 | 3 | 20 | 0 | 1 |
| | $z_j^o$ or $Z_o$ | | 0 | 0 | 0 | 0 | 0 | 0 |
| | $v_j - z_j^o$ | | 8 | 19 | 7 | | 0 | 0 |
| 1 | $v_2 = 19$ | $A_2$ | 3/4 | 1 | 1/4 | 10/4 | 1/4 | 0 |
| | $v_5 = 0$ | $A_5$ | -5/4 | 0 | 9/4 | 50/4 | -3/4 | 1 |
| | $z_j^1$ , $Z_1$ | | 57/4 | 19 | 19/4 | $47\frac{1}{2}$ | 19/4 | 0 |
| | $v_j - z_j^1$ | | -25/4 | 0 | 9/4 | | -19/4 | 0 |
| 2 | $v_2 = 19$ | $A_2$ | 8/9 | 1 | 0 | 10/9 | 1/3 | -1/9 |
| | $v_3 = 7$ | $A_3$ | -5/9 | 0 | 1 | 50/9 | -1/3 | 4/9 |
| | $z_j^2$ , $Z_2$ | | 117/9 | 19 | 7 | 60 | 4 | 1 |
| | $v_j - z_j^2$ | | -5 | 0 | 0 | | -4 | -1 |

TABLE I

Now taking the inner product of the column headed by B and the columns headed by $A_j (j = 1,2,\ldots,5)$, we get (4.24) as follows

$$3A_4 + A_5 = A_1$$

$$4A_4 + 3A_5 = A_2$$

$$A_4 + 3A_5 = A_3$$

$$A_4 + 0 \cdot A_5 = A_4$$

$$0 \cdot A_4 + A_5 = A_5$$

and the equivalent combinations as

$$X_1^0 = \begin{matrix} 3 \\ 1 \end{matrix} \;,\; X_2^0 = \begin{matrix} 4 \\ 3 \end{matrix} \;,\; X_3^0 = \begin{matrix} 1 \\ 3 \end{matrix} \;,\; X_4^0 = \begin{matrix} 1 \\ 0 \end{matrix} \;,\; X_5^0 = \begin{matrix} 0 \\ 1 \end{matrix}$$

This is the application of (3.1) and (3.3). Next we are to apply (3.15). The first two components of the column headed by V for each r represent $v_i$ of the corresponding process vectors in the basis. The third component represents $z_j^r$ or $Z_r$ by taking the inner product of the column v (first two components) and column $A_j$ (first two components), or the column b (first two components) respectively. At step 0, i.e. $r = 0$,

$$z_1^0 = 3v_4 + v_5 = 3 \times 0 + 1 \times 0 = 0$$

and similarly for $z_2^0$, $z_3^0$, $z_4^0$, and $z_5^0$ .

$$Z_o = 10v_4 + 20v_5 = 10 \times 0 + 20 \times 0 = 0$$

which can be found in the row of $z_j^o$ or $Z_o$ .

Then the next row shows (3.15) or (3.16) with which we determine the vector to be introduced, or with which we can know whether the optimal position is reached or not. At step 0 the largest $v_j - z_j^0$ is the one corresponding to $A_2$ . Thus by (3.15) we select $A_2$ to be in new basis.

Next we have to choose a vector to be removed from the initial basis. By applying (3.9) we take the ratios of the elements of column b to the positive elements of column $A_2$ , we select the smallest one as follows

$$\theta = \min_i \left( \frac{x_{i0}^0}{x_{i2}^0} \text{ for all } x_{i2}^0 > 0 \text{ , } i = 4.5 \right)$$

$$= \min \left(\frac{10}{4}, \frac{20}{3}\right) = \frac{10}{4} = \frac{x_{40}^0}{x_{42}^0}$$

Consequently the vector to be removed from $B_o$ is $A_4$ . The new basis then turns out to be

$$B_1 = (A_2 A_5)$$

We replace $A_4$ in column B on Table I with $A_2$ for the next step $r = 1$ .

Now we are to show how we obtain the tables for $r = 1,2,$ Applying (3.6.1) to obtain the new basic feasible solution, we define

$$\left.\begin{aligned} x_{i0}^{r+1} &= x_{i0}^{r} - \frac{x_{so}^{r}}{x_{st}^{r}} x_{it}^{r} \,, && i \neq S \\ x_{so}^{r+1} &= \frac{x_{so}^{r}}{x_{st}^{r}} && i = S \end{aligned}\right\} \tag{4.26.1}$$

Applying (3.6.2) to get the new equivalent combinations, we define

$$\left.\begin{aligned} x_{ij}^{r+1} &= x_{ij}^{r} - \frac{x_{sj}^{r}}{x_{st}^{r}} x_{it}^{r} \,, && i \neq S \\ x_{sj}^{r+1} &= \frac{x_{sj}^{r}}{x_{st}^{r}} && i = S \end{aligned}\right\} \tag{4.26.2}$$

$$j = 1,2,\ldots,n+m$$

$$j \neq t$$

When $j = t$

$$x_{it}^{r+1} = 0 \ , \ i \neq S$$

$$x_{st}^{r+1} = 1 \ , \ i = S \qquad (4.26.3)$$

We must note that all $i$ defined by (4.26.1)(4.26.2) and (4.26.3) for step $r + 1$ follow those for step $r$ . However we should replace $s$ with $t$ for the next table for step $r + 1$ .

To derive (4.26.1)(4.26.2)(4.26.3) for Example 2, let us apply (3.1') and (3.3)

$$x_{40}^{0}A_4 + x_{50}^{0}A_5 = b \qquad (4.27.1)$$

$$x_{41}^{0}A_4 + x_{51}^{0}A_5 = A_1 \qquad (4.27.2)$$

$$x_{42}^{0}A_4 + x_{52}^{0}A_5 = A_2 \qquad (4.27.3)$$

$$x_{43}^{0}A_4 + x_{53}^{0}A_5 = A_3 \qquad (4.27.4)$$

$$x_{44}^{0}A_4 + x_{54}^{0}A_5 = A_4 \qquad (4.27.5)$$

$$x_{45}^{0}A_4 + x_{55}^{0}A_5 = A_5 \qquad (4.27.6)$$

Sincw we are to replace $A_4$ with $A_2$ , to get the new basic feasible solution we can substitute (4.27.3) into (4.27.1) by eliminating $A_4$ . Dividing (4.27.3) by $x_{42}^{0}$ and moving the second term on the left hand side to the right hand side, we get

$$A_4 = \frac{1}{x_{42}^{0}}A_2 - \frac{x_{52}^{0}}{x_{42}^{0}} A_5$$

Substituting in (4.27.1) we get

$$\frac{x_{40}^{0}}{x_{42}^{0}} A_2 + (x_{50}^{0} - \frac{x_{40}^{0}}{x_{42}^{0}} x_{52}^{0}) A_5 = b$$

Thus the new basic feasible solution with respect to the new basis $B_1 = (A_2 A_5)$ is

$$x_{20}^{1} = \frac{x_{40}^{0}}{x_{42}^{0}}$$

$$x_{50}^{1} = x_{50}^{0} - \frac{x_{40}^{0}}{x_{42}^{0}} x_{52}^{0}$$

These are exactly what (4.26.1) indicates, and the numerical values are shown in column b of the second table of Table I. To obtain (4.26.2) for all $j \neq 2$, we can substitute (4.27.3) into (4.27.2) (4.27.4) (4.27.5) (4.27.6) and eliminate $A_4$. The elements in columns $A_1$, $A_3$, $A_4$, and $A_5$ are simply computed according to (4.26.2). We must note that when we substitute (4.27.3) into (4.27.3) itself to eliminate $A_4$, i.e. when $j = 2$, we get

$$\frac{x_{42}^{0}}{x_{42}^{0}} A_2 + (x_{52}^{0} - \frac{x_{42}^{0}}{x_{42}^{0}} x_{52}^{0}) A_5 = A_2$$

or $A_2 + 0 \cdot A_5 = A_2$

i.e. $x_{22}^{1} = 1$ ; $x_{52}^{1} = 0$

which is simply what (4.26.3) indicates, and appears in the column $A_2$ of the second table (i.e. $r = 1$) on Table I.

We must also note that the columns corresponding to the vectors in the new basis are unit vectors, since $x_{22}^1 = 1$ , $x_{52}^1 = 0$ , $x_{25}^1 = 0$ , $x_{55}^1 = 1$ . Thus each time when we introduce a new vector, we get a unit vector. Consequently we can always proceed with identity matrix as basis for any $r$ . Thus with identity matrix as basis we can continue our computation for $r = 2,3,\ldots$ as when $r = 1$ . Our computation will be iterated until $v_j - z_j^r \leqq 0$ for all $j$ .

Our final table shows that the optimal solution is

$$X_0^2 = \begin{bmatrix} x_{20}^2 \\ x_{30}^2 \end{bmatrix} = \begin{bmatrix} 10/9 \\ 50/9 \end{bmatrix}$$

by reading the first two elements in column b where $r = 2$. The maximum value of objective function is 60 by reading the element at column b and row $(z_j^2, Z_2)$ . We also found that $Z_r$ increases from 0 when $r = 0$, to $47\frac{1}{2}$ when $r = 1$, and to 60 when $r = 2$ . This result is identical with that in (4.11.2)

Next we should like to demonstrate the table computation technique by using a larger example. Let us use Example 3. Recall (4.12.1') (4.12.2') and (4.12.3')

$$\text{Maximize} \quad Z = 3x_1 + 4x_2 + x_3 + 7x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 0 \cdot x_7$$

$$\text{Subject to} \quad 8x_1 + 3x_2 + 4x_3 + x_4 + x_5 + 0 \cdot x_6 + 0 \cdot x_7 = 7$$

$$2x_1 + 6x_2 + x_3 + 5x_4 + 0 \cdot x_5 + x_6 + 0 \cdot x_7 = 3$$

$$x_1 + 4x_2 + 5x_3 + 2x_4 + 0 \cdot x_5 + 0 \cdot x_6 + x_7 = 8$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geqq 0$$

We start with the identity matrix including $A_5, A_6, A_7$. At each step we choose a vector with the largest positive $v_j - z_j^r$ to replace a vector with

$$\theta = \min_i \left(\frac{b_i}{a_{ij}}\right) \quad \text{for all positive } a_{ij}$$

For Example 3 we have two steps of iteration, and the results of final table are exactly identical with those obtained in (4.19.1) and (4.19.2). Our table computation is shown as follows

| r | V | B | $A_1$ | $A_2$ | $A_3$ | $A_4$ | b | $A_5$ | $A_6$ | $A_7$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $v_5 = 0$ | $A_5$ | 8 | 3 | 4 | 1 | 7 | 1 | 0 | 0 |
| | $v_6 = 0$ | $A_6$ | 2 | 6 | 1 | 5 | 3 | 0 | 1 | 0 |
| | $v_7 = 0$ | $A_7$ | 1 | 4 | 5 | 2 | 8 | 0 | 0 | 1 |
| | $z_j^o$ , $Z_o$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $V_j - z_j^o$ | | 3 | 4 | 1 | 7 | | 0 | 0 | 0 |
| 1 | $v_5 = 0$ | $A_5$ | 38/5 | 9/5 | 19/5 | 0 | 32/5 | 1 | -1/5 | 0 |
| | $v_4 = 7$ | $A_4$ | 2/5 | 6/5 | 1/5 | 1 | 3/5 | 0 | 1/5 | 0 |
| | $v_7 = 0$ | $A_7$ | 1/5 | 8/5 | 23/5 | 0 | 34/5 | 0 | -2/5 | 1 |
| | $z_j^1$ , $Z_1$ | | 14/5 | 42/5 | 7/5 | 7 | $4\frac{1}{5}$ | 0 | 7/5 | 0 |
| | $V_j - z_j^1$ | | 1/5 | -22/5 | -2/5 | 0 | | 0 | -7/5 | 0 |
| 2 | $v_3 = 3$ | $A_1$ | 1 | 9/38 | 19/38 | 0 | 32/38 | 5/38 | -1/38 | 0 |
| | $v_4 = 7$ | $A_4$ | 0 | 42/38 | 0 | 1 | 10/38 | -2/38 | 8/38 | 0 |
| | $v_7 = 0$ | $A_7$ | 0 | 59/38 | 9/2 | 0 | 252/38 | -1/38 | -15/38 | 1 |
| | $z_j^2$ , $Z_2$ | | 3 | 321/39 | 57/38 | 7 | $4\frac{7}{19}$ | 1/38 | 53/38 | 0 |
| | $V_j - z_j^2$ | | 0 | -169/38 | -1/2 | 0 | | -1/38 | -53/38 | 0 |

TABLE II

So far we have demonstrated the table computation with slack basis as initial basis. However we must note that the slack basis is not always available. We may have an original canonical program which does not have identity matrix in its matrix $A$ . In this case we may introduce some artificial (unit) vectors such that the initial identity matrix is available. There are several methods available in this respect. We shall follow Charnes' Method. [3]

## §3. Artificial Basis as Initial Basis

Let us use the following example. This time it is a minimization problem.

Example 4. [4]

$$\min \quad Z = 2x_1 + x_2 - x_3 - x_4 \tag{4.28.1}$$

$$\text{subject to} \quad x_1 - x_2 + 2x_3 - x_4 = 2$$

$$2x_1 + x_2 - 3x_3 + x_4 = 6 \tag{4.28.2}$$

$$x_1 + x_2 + x_3 + x_4 = 7$$

$$x_1, x_2, x_3, x_4 \geqq 0 \tag{4.28.3}$$

As mentioned in Chapter 1, any minimization problem can be transformed into maximization problem by multiplying the objective function by -1. Thus (4.28.1) turns out to be

$$Z' = -2x_1 - x_2 + x_3 + x_4$$

---

[3] -M Method is originally developed by Charnes. Our discussion follows Hadley, op. cit. pp. 116 - 121.

[4] Example 4 is an exercise in Gass op. cit., p. 81, Exercise 1-C.

where $Z' = -Z$

Now since (4.28.2) is a set of equations, we can introduce vectors in such a way that we can start with an identity matrix as basis. One way out of this difficulty is to add arbitrarily an identity matrix to (4.28.2), and assign a sufficiently small number -M to the corresponding new $v_i$ so that the artificial vectors must disappear from the final basis. Since the added vectors are artificial, we must try to let them leave the basis as soon as possible. For this purpose we can assign such a negative number -M that it appears very large in $v_j - z_j^r$ when we apply (3.15). The absolute value of -M must be assigned very large in such a way that any number associated with it can be neglected. By doing so the non-artificial vectors will be chosen to replace the artificial vectors. Now Example 4 can be rewritten as

$$\max \quad Z' = -2x_1 - x_2 + x_3 + x_4 - Mx_5 - Mx_6 - Mx_7$$

$$\text{subject to } [A,I] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix}$$

$$x_1, x_2, x_3, x_4 \geqq 0$$

where I is identity matrix formed by the artificial vectors.

Initially we shall find $Z'$ is unduly small if the artificial

variables appear in the function. Obviously $Z'$ will not be maximum. When $x_5 = x_6 = x_7 = 0$, any solution will yield a $Z'$ much greater than that when $x_5, x_6, x_7 > 0$.

Even if the solution is not a maximum, $Z'$ will be much greater than that when any one of artificial variables appears, since the absolute value of $M$ is such a large number that any numbers such as $v_i$ $(i = 1,2,3,4)$ can be neglected.

| r | v | B | $A_1$ | $A_2$ | $A_3$ | $A_4$ | b | $A_5$ | $A_6$ | $A_7$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $v_5 = -M$ | $A_5$ | 1 | -1 | 2 | -1 | 2 | 1 | 0 | 0 |
| | $v_6 = -M$ | $A_6$ | 2 | 1 | -3 | 1 | 6 | 0 | 1 | 0 |
| | $v_7 = -M$ | $A_7$ | 1 | 1 | 1 | 1 | 7 | 0 | 0 | 0 |
| | $z_j^o$ , $Z_o$ | | -4M | -M | 0 | -M | -15M | -M | -M | -M |
| | $V_j - z_j^o$ | | -2+4M | -1+M | 1 | 1+M | | 0 | 0 | 0 |
| 1 | $v_1 = -2$ | $A_1$ | 1 | -1 | 2 | -1 | 2 | 1 | 0 | 0 |
| | $v_6 = -M$ | $A_6$ | 0 | 3 | -7 | 3 | 2 | -2 | 1 | 0 |
| | $v_7 = -M$ | $A_7$ | 0 | 2 | -1 | 2 | 5 | -1 | 0 | 1 |
| | $z_j^1$ , $Z_1$ | | -2 | -2-5M | -4+8M | 2-5M | -4-7M | -2+3M | -M | -M |
| | $V_j - z_j^1$ | | 0 | -3+5M | 5-8M | -1+5M | | -4M+2 | 0 | 0 |
| 2 | $v_1 = -2$ | $A_1$ | 1 | 0 | -1/3 | 0 | 8/3 | 1/3 | 1/3 | 0 |
| | $v_4 = 1$ | $A_4$ | 0 | 1 | -7/3 | 1 | 2/3 | -2/3 | 1/3 | 0 |
| | $v_7 = -M$ | $A_7$ | 0 | 0 | 11/3 | 0 | 11/3 | 1/3 | -2/3 | 1 |
| | $z_j^2$ , $Z_2$ | | -2 | 1 | $\frac{-5-11M}{3}$ | 1 | $\frac{-14-11M}{3}$ | $\frac{-M-4}{3}$ | $\frac{-1+2M}{3}$ | -M |
| | $V_j - z_j^2$ | | 0 | -2 | $\frac{8}{3}+\frac{11}{3}M$ | 0 | | $\frac{-2M+4}{3}$ | $\frac{-5M+1}{3}$ | 0 |
| 3 | $v_1 = -2$ | $A_1$ | 1 | 0 | 0 | 0 | 0 | 4/11 | 3/11 | 1/11 |
| | $v_4 = 1$ | $A_4$ | 0 | 1 | 0 | 1 | 3 | -5/11 | -1/11 | 7/11 |
| | $v_3 = 1$ | $A_3$ | 0 | 0 | 1 | 0 | 1 | 1/11 | -2/11 | 3/11 |
| | $z_j^3$ , $Z_3$ | | -2 | 1 | 1 | 1 | -2 | -12/11 | -9/11 | 8/11 |
| | $V_j - z_j^3$ | | 0 | -2 | 0 | 0 | | $-M+\frac{12}{11}$ | $-M+\frac{9}{11}$ | $-M-\frac{8}{11}$ |

TABLE III

On Table III, we can find $v_j - z_j^r$ very large in the non-artificial columns, and very small in the artificial columns. This helps us choose the non-artificial vectors to replace the artificial ones. In addition the value of $Z'_1 = -4 -7M$ is greater than $Z'_0 = -15M$, since $-15M < -7M$, and 4 is negligible compared with $M$. Consequently by reading column b for all $r$, we can find the value of $z'_r$ increases sharply at each $r$. i.e. at each step of iteration, we have

$$-15M < -4-7M < -\frac{14}{3} -3\frac{2}{3}M < -2$$

Finally when $r = 3$, we get $z'_3 = -2$, which is the maximum by the optimality criterion (3.16), since at step 3 the $v_j - z_j^3$ (for all $j = 1,2,\ldots,7$) are non-positive.

Instead of using (3.16) we can also check the optimal position by using the Duality Theorem which was discussed in Chapter 2.

## §4. Alternative Optimality Criterion

First we should like to point out that the values of $z_j^r$ corresponding to the initial basis, i.e. the basis formed by slack or artificial vectors on the final table construct the dual optimal solution. To verify this result we write the equivalent combinations to the initial vectors $A_{n+1}, A_{n+1}, \ldots, A_{n+m}$ as follows

$$X_j^f = B_f^{-1} A_j$$

$$j = n + 1,\ n + 2,\ \ldots,\ n + m$$

f indicates the final basis index number

Putting all $X_j^f$ in one format, we get an $m \times m$ matrix as

$$(X_{n+1}^f X_{n+2}^f \ldots X_{n+m}^f) = B_f^{-1} (A_{n+1} A_{n+2} \ldots A_{n+m})$$

Premultiplying by $V_f'$, we get

$$V_f'(X_{n+1}^f X_{n+2}^f \ldots X_{n+m}^f) = V_f' B_f^{-1} (A_{n+1} A_{n+2} \ldots A_{n+m}) \tag{4.29}$$

where $V_f$ is an $m \times 1$ column vector as defined in (2.23.1)

Transpose (2.23.2)

$$y^{o\prime} = V_f' B_f^{-1}$$

Substituting in (4.29), we get

$$V_f'(X_{n+1}^f X_{n+2}^f \ldots X_{n+m}^f) = y^{o\prime}(A_{n+1} A_{n+2} \ldots A_{n+m}) \tag{4.30}$$

However $(A_{n+1} A_{n+2} \ldots A_{n+m})$ is an identity matrix, since it is the initial basis formed by slack vectors.

Consequently (4.30) turns out to be

$$V_f'(X_{n+1}^f X_{n+2}^f \ldots X_{n+m}^f) = y^{o\prime}$$

$$\text{or} \quad y^o = \begin{bmatrix} V_f' X_{n+1}^f \\ V_f' X_{n+2}^f \\ \vdots \\ V_f' X_{n+m}^f \end{bmatrix} \tag{4.31}$$

Recalling (3.14), the right hand side of (4.31) simply indicates $z_j^r$, $j = n+1, n+2, \ldots, n+m$, at the final step. The left hand side is

the optimal solution to the dual problem by (2.27) and (2.28).

Let us then check our results of Examples 2,3, and 4. On Table I, the optimal $Z$ of primal problem is 60, while the optimal $W$ of the dual is

$$b'y^o = b_1y_1^o + b_2y_2^o = 10 \times 4 + 20 \times 1 = 60$$

$$\max Z(x) = \min W(y)$$

On Table II, the max $Z(x)$ of primal problem is $4\frac{7}{19}$, while the min $W(y)$ of the dual is

$$b'y^o = b_1y_1^o + b_2y_2^o + b_3y_3^o = 7 \times \frac{1}{38} + 3x\frac{53}{38} + 8 \times 0 = 4\frac{7}{19}$$

$$\max Z(x) = \min W(y)$$

On Table III, max $Z(x) = -2$, while

$$\min W(y) = 2x\frac{-12}{11} + 6x\frac{-9}{11} + 7 \times \frac{8}{11} = -2$$

$$\max Z(x) = \min W(y)$$

We must note that the optimal solution to the dual problem of Example 4 is

$$y_1^o = \frac{-12}{11}, \quad y_2^o = \frac{-9}{11}, \quad y_3^o = \frac{8}{11}$$

Since Example 4 was a minimization problem, and we multiplied the objective function by -1 to transform it into maximization problem, we have to multiply the optimal $Z$ by -1 again to get the true value. Therefore the true value of objective function turns out to be 2, and the dual optimal

solution is $$y_1^o = \frac{12}{11}\ , \ y_2^o = \frac{9}{11}\ , \ y_3^o = \frac{-8}{11}$$

Note $y_3^o$ is negative. Since Example 4 is an original canonical linear program, the dual solution is not constrained to be non-negative. This verifies our assertion in Chapter 1.

# CHAPTER 5

## GEOMETRIC INTERPRETATION

Recall our Example 2.

$$\text{maximize} \quad Z = 8x_1 + 19x_2 + 7x_3 + 0.x_4 + 0.x_5 \tag{5.1.1}$$

$$\text{subject to} \quad \begin{aligned} 3x_1 + 4x_2 + x_3 + x_4 + 0.x_5 &= 10 \\ x_1 + 3x_2 + 3x_3 + 0.x_4 + x_5 &= 20 \end{aligned} \tag{5.1.2}$$

$$x_1, x_2, x_3, x_4, x_5 \geqq 0 \tag{5.1.3}$$

Before we start our geometric interpretation, we must point out that in this chapter we shall use point and vector interchangeably. Point is identical with vector in geometric sense.

Since $A^*$ is a $2 \times 5$ matrix, we consider a 2-dimensional vector space, $V_2$ , in which there are six vectors as

$$A_1 = \begin{bmatrix}3\\1\end{bmatrix}, \quad A_2 = \begin{bmatrix}4\\3\end{bmatrix}, \quad A_3 = \begin{bmatrix}1\\3\end{bmatrix}$$

$$A_4 = \begin{bmatrix}1\\0\end{bmatrix}, \quad A_5 = \begin{bmatrix}0\\1\end{bmatrix}, \quad b = \begin{bmatrix}10\\20\end{bmatrix} .$$

The equations (5.1.2) simply state that a 5-dimensional vector $x^* = (x_1, x_2, x_3, x_4, x_5)'$ in vector space $V_5$ is mapped through $A^* = (A_1\ A_2\ A_3\ A_4\ A_5)$ into a 2-dimensional vector $b = (10, 20)'$ in $V_2$ . Define a subspace of $V_5$ as $S_5 \subset V_5$ . Then we are going to find a non-negative vector $X_o^r$ in $S_5$ such that $Z$ is maximized, and $B_r X_o^r = b$ is satisfied, where

$X_o^r$ is a $2 \times 1$ column vector as defined in Chapter 2, and $B_r$ a $2 \times 2$ matrix containing 2 vectors in $V_2$. However, as mentioned before, $X_o^r$ can be considered as a $5 \times 1$ column vector with three elements being zero, and $B_r$ can be considered as a $2 \times 5$ matrix with 3 columns being null vectors.

In other words, we are to find a point $X_o^r$ in $S_5$ such that the vector b lies in the convex cone formed by 2 vectors in $V_2$, and such that Z is maximum. Define $C_2$ as a 2-dimensional cone. By a cone we mean that a point, say $A_i$ $(i = 1, 2, ..., 5)$, is in $C_2$, then $\lambda_i A_i$ is also in $C_2$ for any scalar $\lambda_i \geqq 0$. By a convex cone we mean that if $A_j$ and $\lambda_i A_i$ are in $C_2$, then $A_j + \lambda_i A_i$ is also in $C_2$. For example, let us see Fig. IV. The point $A_1$ is a vector from the origin pointing at coordinates (3,1), point $A_2$ at (4,3), point $A_3$ at (1,3), point $A_4$ at (1,0), point $A_5$ at (0,1). The cone of $A_1$ is the ray from the origin all the way out through point (3,1). Thus in Fig. IV, $A_1$, $2A_1$, $3A_1$, ... are also in the cone. Similarly for the other vectors. Now any non-negative linear combination of 2 vectors in $V_2$ (3 in $V_3$, 4 in $V_4$, etc.) lies between the two vectors. Thus the convex cone formed by two independent vectors is simply the space between the two vectors. On Fig. IV, $A_1$ is in the convex cone formed by $A_2$ and $A_4$, b is in the convex cone of $A_2$ and $A_3$, etc.

We have known in Chapter 3, there are ten possible bases for Example 2. On Fig. IV any two vectors form a basis, since any two from the five vectors in $V_2$ are independent with each other, i.e. $A_1 \neq \lambda_2 A_2$, $A_1 \neq \lambda_3 A_3$, etc., where $\lambda_2$ and $\lambda_3$ are scalars. We can

find all the basic feasible solutions from Fig. IV. As long as vector $b$ lies in the convex cone of any two $A_i$, $(i = 1, 2, \ldots, 5)$, there exists a basic feasible solution. Looking at vectors $A_1$ and $A_2$, i.e. $B_1$ defined in (4.2), vector $b$ does not lie between $A_1$ and $A_2$, therefore there is no basic feasible solution with respect to $B_1$. This result is identical with our result in (4.3.2). Intuitively between $A_1$ & $A_3$, $A_1$ & $A_5$, $A_2$ & $A_3$, $A_2$ & $A_5$, $A_3$ & $A_4$, and $A_4$ & $A_5$, there lies the vector $b$. Consequently, there exist basic feasible solutions involving these bases, $B_2 = (A_1A_3)$, $B_4 = (A_1A_5)$, $B_5 = (A_2A_3)$, $B_7 = (A_2A_5)$, $B_8 = (A_3A_4)$, and $B_{10} = (A_4A_5)$ as defined in (4.2). The last basis $B_{10}$ contains $A_4$ and $A_5$ which are slack vectors. Thus, as mentioned before, it is meaningless to form a basis with all component vectors being slack. Therefore we can conclude that, by looking at Fig. IV, the basic feasible solutions to be found involve only $B_2$, $B_4$, $B_5$, $B_7$, and $B_8$. This conclusion is exactly the same as in (4.3.24).

Since there are two vector spaces, $V_2$ and $V_5$ [1], in regard to the linear transformation (or mapping), $A^*x^* = b$ (5.1.2), we can solve this program geometrically either in terms of $V_2$, or in terms of $V_5$.

Geometric solution in terms of 5-dimensional space $V_5$ was shown in Chapter 2 by Fig. III. Now we shall solve this linear program by Method A geometrically in terms of $V_2$. Since intuitively we have known there are only five basic feasible solutions to be checked. (We must note that the possible steps of iteration are still ten). Let us

[1] Hadley calls $V_2$ requirements space, and $V_5$ solutions space. Hadley, Linear Programming, p. 158, and p. 162.

start with $B_2 = (A_1 A_3)$ . We are to find a set of non-negative scalars $\lambda_1^2$ and $\lambda_3^2$ on Fig. IV such that the condition

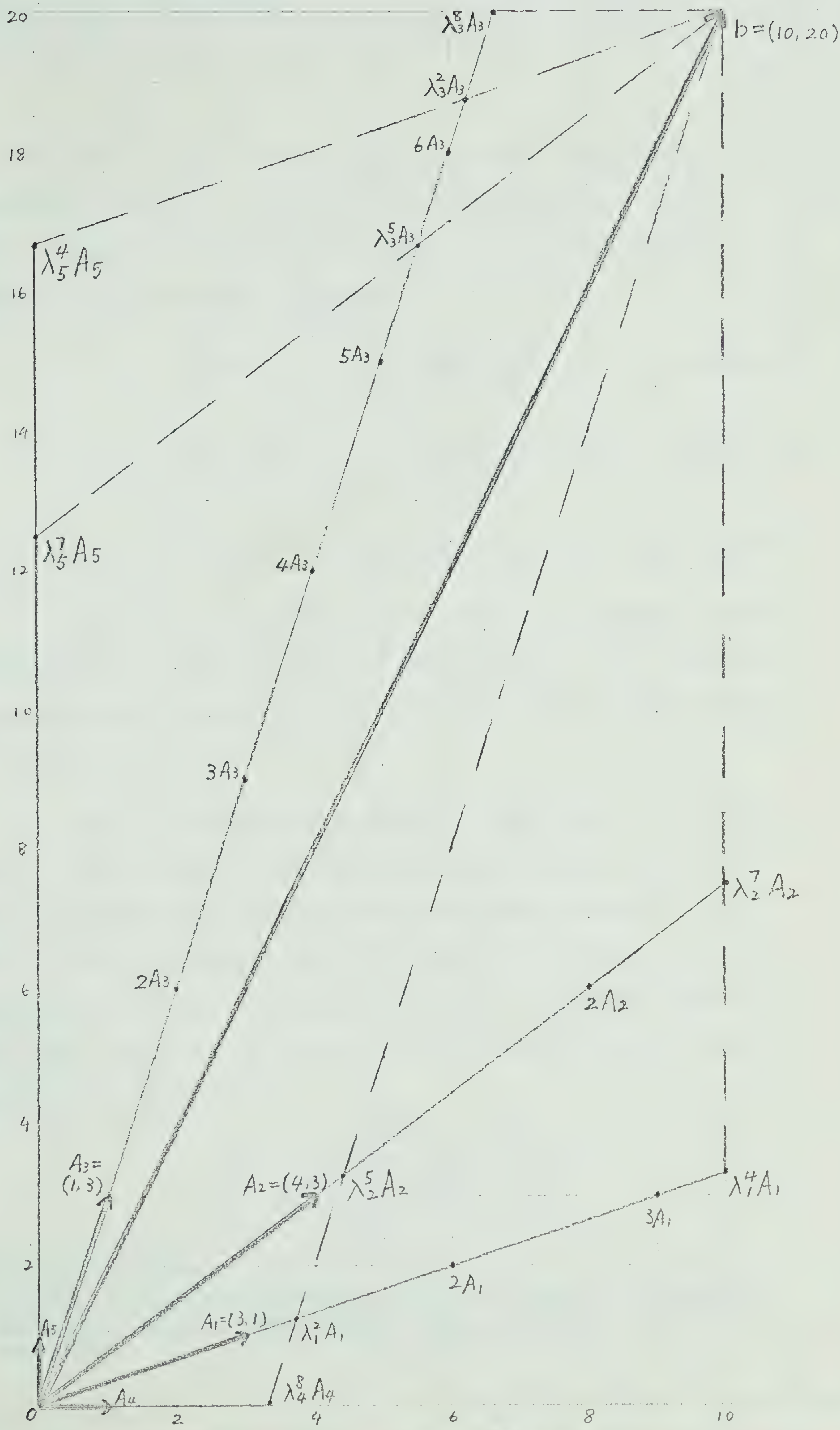
$$\lambda_1^2 A_1 + \lambda_3^2 A_3 = b \tag{5.2}$$

is satisfied. In other words, we are to find a linear combination of $A_1$ and $A_3$ with coefficients $\lambda_1^2$ , $\lambda_3^2 \geqq 0$ such that vector b is in the convex cone of $A_1$ and $A_3$ .

We can use the Parallelogram Law. Form a parallelogram with the ray between the origin and the point $b = (10,20)$ as diagonal, and draw a line from b extending downward to the left, parallel to the ray from the origin through point $A_1 = (3,1)$, then draw a line from b downward parallel to the ray from the origin through point $A_3 = (1,3)$ . Thus we can find two points of intersections at $\lambda_1^2 A_1$ and $\lambda_3^2 A_3$ . i.e. at the points $\lambda_1^2$ times of $A_1$ and $\lambda_3^2$ times of $A_3$ , we can find a unique, positive (non-degenerate) linear combination of $A_1$ and $A_3$ , such that (5.2) is satisfied. Thus the parallelogram turns out to be $0\ \lambda_1^2 A_1\ b\ \lambda_3^2 A_3$ .

By reading Fig. IV, the point $\lambda_1^2 A_1$ is 1 1/4 times of $A_1$ , and $\lambda_3^2 A_3$ is 6 1/4 times of $A_3$ . This result is identical with (4.3.4). Consequently $\lambda_1^2$ and $\lambda_3^2$ can be identified as $x_{10}^2$ and $x_{30}^2$ respectively.

The next basic feasible solution involves $B_4 = (A_1 A_5)$ . Similarly as above we can form a parallelogram $0\lambda_1^4 A_1\ b\ \lambda_5^4 A_5$ where points $\lambda_1^4 A_1$ and $\lambda_5^4 A_5$ are the intersections we want. Point $\lambda_1^4 A_1$



20
18
16
14
12
10
8
6
4
2
0
2
4
6
8
10
$\lambda_3^8 A_3$
$b=(10, 20)$
$\lambda_3^2 A_3$
$6A_3$
$\lambda_3^5 A_3$
$\lambda_5^4 A_5$
$5A_3$
$\lambda_5^7 A_5$
$4A_3$
$3A_3$
$\lambda_2^7 A_2$
$2A_3$
$2A_2$
$A_3 = (1, 3)$
$A_2 = (4, 3)$
$\lambda_2^5 A_2$
$\lambda_1^4 A_1$
$3A_1$
$2A_1$
$A_1 = (3, 1)$
$\lambda_1^2 A_1$
$A_5$
$\lambda_4^8 A_4$
$A_4$


Fig. IV

is 3 1/3 times of $A_1$, and point $\lambda_5^4 A_5$ is 16 2/3 times of $A_5$ intuitively on Fig. IV. This result is also identical with (4.3.9). Thus $\lambda_1^4$ and $\lambda_5^4$ are identified as $x_{10}^4$ and $x_{50}^4$ respectively. Similarly for $B_5$, $B_7$ and $B_8$ we find

$$\lambda_2^5 = 1\tfrac{1}{9} = x_{20}^5 \ ; \ \lambda_3^5 = 5\tfrac{5}{9} = x_{30}^5 \qquad \text{by (4.3.12)}$$

$$\lambda_2^7 = 2\tfrac{1}{2} = x_{20}^7 \ ; \ \lambda_5^7 = 12\tfrac{1}{2} = x_{50}^7 \qquad \text{by (4.3.17)}$$

$$\lambda_3^8 = 6\tfrac{2}{3} = x_{30}^8 \ ; \ \lambda_4^8 = 3\tfrac{1}{3} = x_{40}^8 \qquad \text{by (4.3.20)}$$

Finally we can find the maximum value of the objective function by multiplying the above solutions by appropriate $v_i$, then choose the largest product. We find $Z_5 = 60$ is the max. This is identical with our result in (4.3.24).

Now let us discuss the geometric interpretation of the simplex method. The mechanism of simplex method mainly involves obtaining equivalent combinations, applying (3.15) and (3.16) to obtain $v_j - z_j^r$, etc. All these procedures involve the values of $v_i, z_j^r$ and $Z_r$. Consequently we shall use a 3-dimensional diagram by adding one dimension of the values $v_i, z_j^r$ and $Z_r$ to the above $V_2$. We can rewrite (5.1.1), and (5.1.2) as follows [2]

$$\begin{bmatrix} 3 & 4 & 1 & 1 & 0 \\ 1 & 3 & 3 & 0 & 1 \\ 8 & 19 & 7 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ Z \end{bmatrix} \tag{5.3}$$

[2] This kind of expression is very common in the theory of the simplex method. We follow partially Hadley, Gass, and Charnes, Cooper and Henderson.

$$\text{or} \quad \begin{matrix} A \\ v' \end{matrix} \; x = \begin{matrix} b \\ Z \end{matrix} \qquad (5.3')$$

For each possible basis, we can write the basic equations as follows

$$\begin{bmatrix} B_r \\ v' \end{bmatrix} X_o^r = b^r \qquad (5.4)$$

where $b^r = (b \; Z_r)'$, an $(m+1) \times 1$ column vector, and $r$ still represents the index number of the basis.

(5.4) is equivalent to (3.1) except the fact that the vector space $V_m$ now becomes $V_{m+1}$ .

Rewrite (3.3) in terms of the new vector space $V_{m+1}$ as follows

$$\begin{bmatrix} B_r \\ v' \end{bmatrix} X_j^r = \begin{bmatrix} A_j \\ z_j \end{bmatrix} \qquad (5.5)$$

for all $A_j$ not in $B_r$

Now let us follow the procedures in Chapter 4 closely from (4.4) through (4.11.2) . Thus we start with a basis $B_2 = (A_1 A_3)$ as defined by (4.2), and rewrite (4.4) and (4.6) by applying (5.4) and (5.5) respectively as follows:

$$x_{10}^2 \bar{A}_1 + x_{30}^2 \bar{A}_3 = b^2 \qquad (5.6)$$

$$\text{where} \quad \bar{A}_1 = \begin{bmatrix} A_1 \\ v_1 \end{bmatrix}, \; \bar{A}_3 = \begin{bmatrix} A_3 \\ v_3 \end{bmatrix}, \; b^2 = \begin{bmatrix} b \\ Z_2 \end{bmatrix}$$

To avoid confusion we use the index number $r$ consistent with that defined in (4.2) for our purpose of interpreting Example 2.

and

$$x_{12}^{2}\bar{A}_1 + x_{32}^{2}\bar{A}_3 = A_2^2$$

$$x_{14}^{2}\bar{A}_1 + x_{34}^{2}\bar{A}_3 = A_4^2 \qquad (5.7)$$

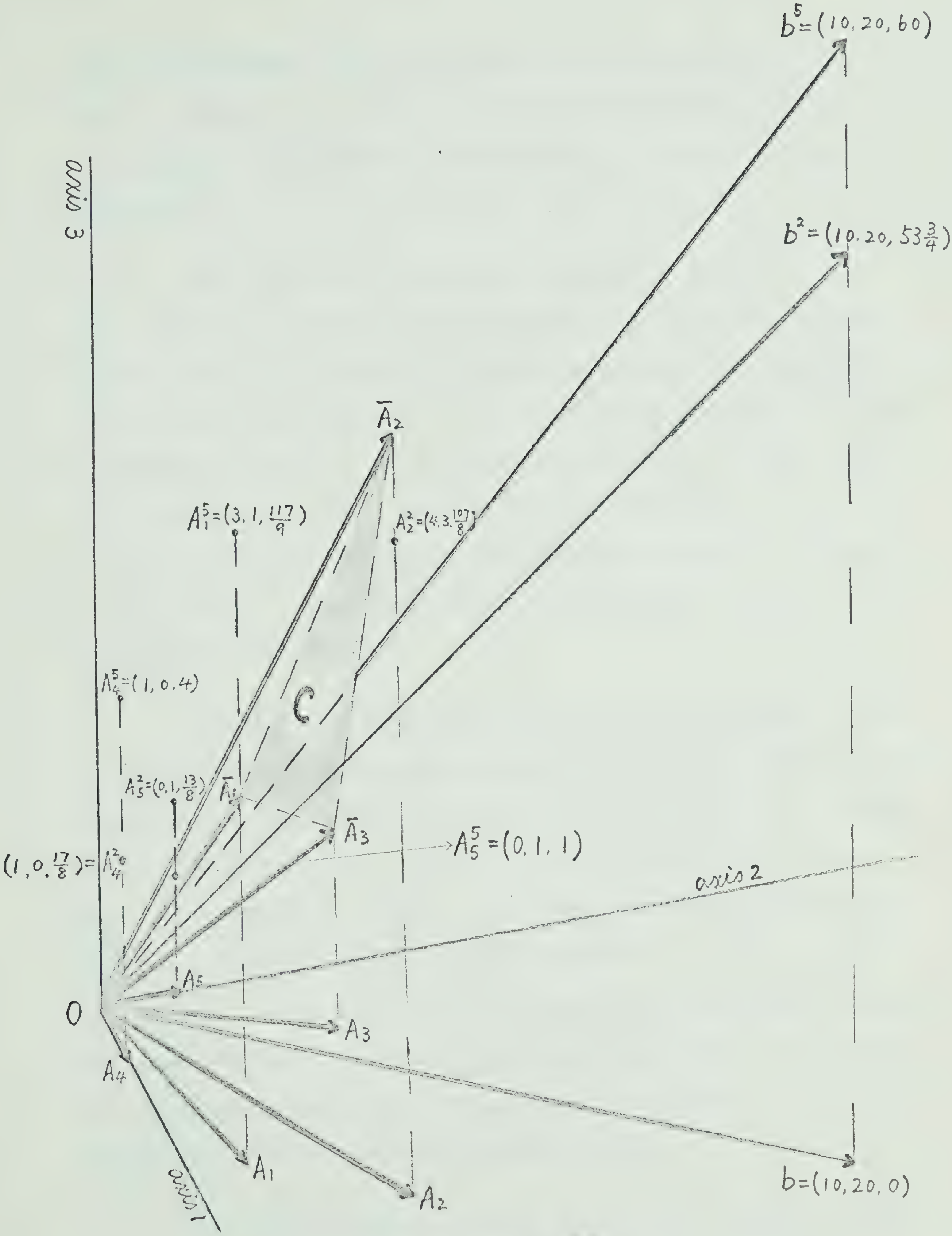
$$x_{15}^{2}\bar{A}_1 + x_{35}^{2}\bar{A}_3 = A_5^2$$

$$\text{where} \quad A_j^r = \begin{bmatrix} A_j \\ z_j^r \end{bmatrix} , \quad \begin{matrix} j \neq 1,\ 3 \\ r = 2\ . \end{matrix} \qquad (5.7.1)$$

Now we can express (5.6) and (5.7) on a 3-dimensional diagram as Fig. V. On Fig. V the third axis represents $v_i$ , $z_j^r$ , and $Z_r$ , while the first and second axes remain the same as on Fig. IV. However we must note that all the axes still represent pure number.

On Fig. V, C , the polyhedron $(0\ \bar{A}_1\ \bar{A}_2\ \bar{A}_3)$ , is the convex cone formed by $\bar{A}_1$ , $\bar{A}_2$ , and $\bar{A}_3$ . We must note that the convex polyhedral cone C is not restricted to that shown on Fig. V. It must include the extension straight out all the way beyond that shown on Fig. V. In other words C contains not only the linear combinations of $A_i$ with coefficients $\lambda_i$ , $0 \leqq \lambda_i \leqq 1$ , it may also contain all the linear combinations of $A_i$ with $\lambda_i \geqq 0$ . However, for our purpose of intuitive interpretation it is better to draw C as shown on Fig. V (i.e. $0 \leqq \lambda_i \leqq 1$) . In addition, other convex cones can also be formed by other non-negative combinations of any three fectors from $\bar{A}_i$ (i = 1, 2, ..., 5) .

On Fig. V, $A_i$ are simply the projections of $\bar{A}_i$ or $A_j^r$ . We must note that $A_i$ here are 3-dimensional vectors with the third components being zero, and that $\bar{A}_i$ and $A_j^r$ differ from $A_i$ only in



$b^5 = (10, 20, 60)$
$b^2 = (10, 20, 53\frac{3}{4})$
axis 3
$\bar{A}_2$
$A_1^5 = (3, 1, \frac{117}{9})$
$A_2^2 = (4, 3, \frac{107}{8})$
$A_4^5 = (1, 0, 4)$
C
$A_5^2 = (0, 1, \frac{13}{8})$
$\bar{A}_1$
$\bar{A}_3$
$A_5^5 = (0, 1, 1)$
$(1, 0, \frac{17}{8}) = A_4^2$
axis 2
$A_5$
O
$A_3$
$A_4$
$A_1$
$A_2$
axis 1
$b = (10, 20, 0)$


FIG V

the third components. On Fig. V, the vertical lines joining $\overline{A}_i$ and $A_i$ represent $v_i$ , while the vertical lines joining $A_j^r$ and $A_j$ represent $z_j^r$ . Similarly the vertical line joining $b^r$ and $b$ represents $Z_r$ .

Now we are ready to interpret Example 2. We have started with $B_2=(A_2A_3)$ . Suppose we have found the basic feasible solution $x_o^2$ as in (4.5.1). Then in the 3-dimensional vector space, one face of $C$ formed by $\overline{A}_1$ and $\overline{A}_3$ , will intersect the vertical line through $b$ upward at a point $b^2$ . This point determines $Z_2 = 53\ 3/4$ which is the result in (4.5.2). Thus the 3-dimensional vector $b^2 = (10, 20, 53\ 3/4)$ must have passed along one face of $C$ formed by $\overline{A}_1$ and $\overline{A}_3$ . This means that the basic solution obtained is non-negative, i.e. feasible.

The next step is to select a vector not in the initial basis $B_2$ to be introduced into the new basis. Suppose we have applied (3.15) and found all $z_j^r$ , $j \neq 1,3$ as shown in (4.7.1), (4.7.2) and (4.7.3). We can find all $z_j^r$ , $j \neq 1, 3$ , in the third coordinates of points $A_2^2$ , $A_4^2$ , and $A_5^2$ on Fig. V . The vertical distances between $A_2^2$ and $\overline{A}_2$ , $A_4^2$ and $\overline{A}_4$ , $A_5^2$ and $\overline{A}_5$ represent $v_2 - z_2^2$ , $v_4 - z_4^2$ , and $v_5 - z_5^2$ . We must note that the term vertical "distances" between two points $\overline{A}_j$ and $A_j^r$ is not used in Euclidean sense, it is simply the difference between the third components of $\overline{A}_j$ and $A_j^r$ . Consequently we do not deal with absolute value.

Now $\overline{A}_4$ and $\overline{A}_5$ are simply $A_4$ and $A_5$ themselves, since

the third components of $\overline{A}_4$ and $\overline{A}_5$ are zeros, i.e. $v_4 = v_5 = 0$ . Thus the vertical distances between $\overline{A}_4$ and $A_4^2$ , $\overline{A}_5$ and $A_5^2$ are negative. The only positive $v_j - z_j^2$ is $v_2 - z_2^2$ , represented by the distance between $\overline{A}_2$ and $A_2^2$ . Therefore, the vector not in $B_2$ to be introduced into the new basis is $A_2$ .

Next we are to determine the vector in $B_2$ to be replaced with $A_2$ . Suppose we have applied (3.9) and found a $\theta$ as shown in (4.8). $\theta$ is found such that the new basic solution is feasible. In our geometric term, $\theta$ is found such that the vector b lies between $A_2$ and one vector from $A_1$ and $A_3$ . Intuitively b does not lie between $A_1$ and $A_2$ , while does lies between $A_3$ and $A_2$ . Therefore the vector to be removed from the initial basis $B_2$ is $A_1$ . Thus the new basis turns out to be $B_5 = (A_2A_3)$ .

Now suppose we have found the basic feasible solution $X_0^5$ as shown in (4.11.1). Then in terms of our 3-dimensional vector space, the face of C formed by $\overline{A}_2$ and $\overline{A}_3$ , $0\,\overline{A}_2\overline{A}_3$ , will intersect the vertical line through b upward at $b^5$ which determines $Z_5 = 60$ . This result is as shown in (4.11.2). The point $b^5$ must be higher than point $b^2$ , since we have introduced a new vector $A_2$ , with $v_2 - z_2^2 > 0$ , into the new basis in such a way that $\theta(v_2 - z_2^2) > 0$ by (3.13). In fact $Z_5 = 60 > Z_2 = 53\ 3/4$ . Thus the vector $b^5 = (10, 20, 60)$ must have passed through the face of C formed by $\overline{A}_2$ and $\overline{A}_3$ , i.e. $0\,\overline{A}_2\overline{A}_3$ . This indicates that $b^5$ is a linear combination of $\overline{A}_2$ and $\overline{A}_3$ with coefficients $X_0^5 > 0$ , i.e. feasible.

Remember we are considering non-degenerate case, otherwise it is non-negative.

Next suppose we have applied (3.3) again, solved (4.9), applied (3.15) and found $z_j^5$ and $v_j - z_j^r$, $j \neq 2, 3$, as shown in (4.10.1), (4.10.2), and (4.10.3). Then again intuitively on Fig. V, the vertical distances between $\overline{A}_j$ and $A_j^5$ again represent (3.15). This time we can find that all the points $A_j^5$, $j \neq 2, 3$, are above the points $\overline{A}_j$, $j \neq 2, 3$, i.e.

$$A_1^5 > \overline{A}_1 \quad , \quad A_4^5 > \overline{A}_4 \ (= A_4) \ , \quad A_5^5 > \overline{A}_5 \ (= A_5)$$

Our optimality criterion (3.16) is satisfied. The point $b^5$ is the highest point on the vertical line through $b$ upward. This linear program is solved.

# CHAPTER 6

## ECONOMIC INTERPRETATION

Now let us interpret the preceding analysis in economic sense. Consider a linear program of production. Recall (1.1.1'), (1.1.2'), and (1.1.3') again. We can consider a case where a firm has $n$ processes of production $A_1, A_2, \ldots, A_n$, and each process produces one product with $m$ limited resources $b_1, b_2, \ldots, b_m$ available. The firm is to determine a level of activities of production $x_1, x_2, \ldots, x_n$ such that the total revenue $v_1x_1 + v_2x_2 + \ldots + v_nx_n$ is maximum, where $v_j$ is the revenue of unit level of production activity $x_j$. The firm also owns $m$ slack processes $A_{n+1}, A_{n+2}, \ldots, A_{n+m}$. This means that the firm can decide to produce nothing by leaving the processes unused, if it is not profitable to engage in production. The component of $A_j$ $(j = 1, 2, \ldots, n)$, $a_{ij}$, can be interpreted as the quantity of resource $b_i$ required to operate one unit level of process $A_j$. It is determined by technology.

The problem is to find the maximum total revenue. We can solve by either Method A, or Simplex Method. If we use Method A, there may be

$$\binom{n+m}{n} \text{ trials .}$$

The firm can select one set of activity levels which yields the largest total revenue. However as mentioned before, this method is not efficient, though good for understanding.

Now the Simplex Method is employed. Suppose the firm starts programming with the first $m$ processes in operation. The firm is to choose one process not in operation to replace a process in operation in such a way that the total revenue is increased. Suppose the first basic solution is feasible, i.e. the first set of activities $x_j$ ( $j = 1, 2, \ldots, m$) is not at negative level, and is technologically feasible. Then by applying (3.3) and (3.15), the firm can choose a process to be operated.

Application of (3.3) is simply to find a combination of activities of the initial processes such that it is equivalent to one unit level of activity of the new process [1]. In other words when the new process is introduced into operation, there will be a change in the level of activities of the old processes. This change when one unit of level of activity of the new process is introduced can be calculated.

Now suppose the idle process to be introduced is $A_t$, then the revenue of unit activity of $A_t$ is $V_t$. On the other hand the value of equivalent combination of activities of processes in operation is

$$z_t^0 = \sum_{i=1}^{m} v_i x_{it}^0 \,, \quad r = 0$$

This is what (3.14) indicates. $z_t^0$ can be interpreted as the cost of introducing a new process.

[1] This is called equivalent combination throughout our discussion of the theory of the simplex method. Dorfman, Samuelson, and Solow, op. cit., p. 150.

Next the firm is to compare $v_t$ and $z_t^0$, i.e. to compare the revenue contributed and the cost incurred by introduction of one unit of process $A_t$. If $v_t > z_t^0$, or $v_t - z_t^0 > 0$, then it is profitable to introduce the idle process $A_t$ into operation. Thus $v_t - z_t^0$ can be considered as the profit contributed by introducing one unit of process $A_t$. However, how did the firm choose $A_t$. At random? No, definitely not. The firm has to compare profits $v_j - z_j^r$ for all idle processes $A_j$ $(j > m)$. (We must note that in table computation we check all $v_j - z_j^r$ for all $j = (1, 2, \ldots, n + m)$. The process to be introduced is then the process with the largest positive $v_j - z_j^r$ among all idle processes $A_j$. In other words, the profit contributed by one unit activity of $A_t$ is the largest among all idle processes. If $v_t - z_t^0 \leqq 0$, then $A_t$ will not be used, since the introduction of $A_t$ will bring the firm loss.

Now the firm is to replace a process in operation with $A_t$. The application of (3.9) is simply to determine the new operating processes such that the new level of activities of production is technologically feasible and non-negative. To interpret this step, let us look at Table II. At step 0, the process to be introduced is $A_4$, and the process to be removed is $A_6$ as circled. Suppose the firm uses only process $A_4$. Then the firm at most can produce 7/1 units of commodity 4 with 7 units of resource $b_1$, or 3/5 unit of commodity 4 with 3 units of resource $b_2$, or 8/2 units of commodity 4 with 8 units of resource $b_3$. In order to make the production of commodity 4 technologically feasible, the firm must

choose 3/5 unit of commodity 4 . i.e. the technological requirements for all resources $b_1$ , $b_2$ , $b_3$ must be satisfied simultaneously. If the firm decides to produce 7/1 units of commodity 4 , resource $b_1$ can satisfy the decision, while resources $b_2$ and $b_3$ can not, since with $b_2$ only 3/5 unit of commodity 4 can be produced, and with $b_3$ only 8/2 units of commodity 4 can be produced. Then the process to be removed is $A_6$ which corresponding 3/5 on the table.

However we must note that at the steps after $r = 0$ , this interpretation will not work, since at step 0 the table represents the resource vector and the coefficient vectors, and at the same time represents the basic solution and equivalent combinations. While at the steps after $r = 0$ , the tables represent equivalent combinations and basic solutions only. Nevertheless, for our purpose of explanation, it does no harm to interpret as at step 0 .

The steps of iteration continue until all the $v_j - z_j^r \leqq 0$ . In other words, the firm keeps on replacing a process in operation with a most profitable idle process until no idle process can be found profitable. This is simply what the optimality criterion indicates.

Next let us interpret the dual program. Remember that the coefficient matrix of (1.2.2) is transpose of $A$ . Let us look at the constraints (1.2.2') , $a_{ji}$ , $(i = 1, \ldots, m)$ are the quantities of all resources $b_i$ $(i = 1, \ldots, m)$ required to produce one unit of commodity $j$ . The constraint for commodity $j$ is

$$\sum_{i=1}^{m} a_{ji}\, y_i \geqq v_j \tag{6.1}$$

The problem is to find a set of $y_i$ $(i = 1, \ldots, m)$ such that (1.2.1'), (1.2.2') and (1.2.3') are satisfied.

The set of variables $y_i$ $(i = 1, \ldots, m)$ is subject to interpretation. The left hand side of (6.1) shows some linear combination of the quantities of $m$ resources needed to operate one unit level of process $A_j$, while the right hand side is the revenue contributed by one unit level of process $A_j$. To have the values of both sides consistent, it is plausible to interpret $y_i$ as some sort of price [2]. By judging (1.2.1'), the value of the objective function, it is appropriate to interpret $y_i$ as the costs of resources $b_i$ $(i = 1, 2, \ldots, m)$. Then we can interpret the dual program as follows

> The firm is to find a set of costs (or shadow prices) of resources (or inputs) such that the total cost of production is minimized subject to the constraints (1.2.2') and (1.2.3').

The interpretation of (1.2.2') or (6.1) for all $j = 1, 2, \ldots, n$ is not an easy one. Some theorists of linear programming leave this task untouched. Many have tried this task, but still left some controversial points. K. Lancaster interpreted (6.1) as follows

> "The constraints of the dual express the fact that if the value of inputs incorporated into a product were less than the

---

[2] Dorfman, Samuelson, and Solow, op. cit., p. 43.

price of the product, it would be more profitable to produce and sell the output than to sell the resources." [3]

This interpretation would be plausible, if the firm had two alternative jobs, one to use the resources to engage in production, the other to stop production and to sell the resources. However, if this is the case, then the dual problem is not the dual of the primal problem. It is an independent minimization problem. In other words, the dual problem and the primal problem should exist simultaneously, not alternatively. If we follow Lancaster, and if the firm takes the first job, i.e. use resources and produce commodities, then there does not exist an optimal solution. According to his argument cited above, it seems that the firm will produce and sell its products, if the dual constraints are not satisfied. However, we argue that the dual constraints and the primal ones must be satisfied simultaneously even if the optimal position is not yet reached. The two problems are simply two faces of one problem. Otherwise the dual problem would be an independent primal, not a dual, problem. Similarly, the same argument applies to the case where the firm takes the second job, i.e. to stop production and to sell its resources.

Now we would like to try our own interpretation of (6.1). Our primal constraints state that the utilization of resources in production should not exceed the limited amount of resources available.

---

[3] Lancaster, op. cit., p. 35. I am responsible for criticizing Lancaster. There are many different interpretations of the dual program. e.g. Baumol interpreted $y$ as the accounting prices of the resources, and the dual constraints as a statement of no-accounting-profit requirement, William J. Baumol, Economic Theory and Operations Analysis, second edition, pp. 107 - 110.

But the dual constraints state that the cost of resources (inputs) used in producing one kind of commodity must not be less than the price of the commodity. This seems paradoxical at first sight. However, mathematically, if a maximization problem is bounded, it is bounded from above in the usual case. On the other hand a minimization problem is a problem bounded from below. Now economically, a profit-maximizing firm must be subject to some limits. The firm can increase cost or loss freely without limitation. Therefore, when the firm is considering a production program, it can either expect low profits among the $\binom{n+m}{m}$ alternative programs, then the job of programming is to find one program which yields the largest profit, or expect high costs, then find one program with lowest cost. Thus $y_i$ can be interpreted as expected price of $b_i$. Finally according to the Duality Theorem, max $Z(x) =$ min $W(y)$ at optimum. In other words, at optimum total revenue is equal to total cost, or zero profit. This is simply a state of competitive equilibrium.

So far it seems that the firm incurs losses all the time except when optimal position is reached. However, in production programming (linear) there is no time dimension included. The programming is static in the sense that it does not involve any adjustment process. We must distinguish the dynamic mechanism of market adjustment from the programming iteration for searching for optimal solution in the static sense.

Thus we can interpret our production programming as that by assuming competitive equilibrium, the firm is to find an optimal level

of production activities among the $\binom{n+m}{m}$ alternatives defined in (2.13), such that it satisfies the condition of competitive equilibrium. The firm would not incur loss in actual production.

Finally we can use the Duality Theorem and the Existence Theorem to express a state of competitive equilibrium. i.e. the Duality Theorem represents a state of zero profit

$$v'x = b'y$$

Total revenue = Total cost

In regard to the Existence Theorem, we can interpret [4]

(a) If $x > 0$ , then $A'y = 0$

$A'y > v$ then $x = 0$

as that production at positive level will just cover the cost of production, while no production will be carried on at a loss. Further we can interpret

(b) If $Ax < b$ , then $y = 0$

and $y > 0$ , then $Ax = b$

as that the resources not used up are free goods, and the resources have positive prices when all the resources are fully employed.

This simply indicates a state of competitive equilibrium.

---

[4] Lancaster, op. cit., pp. 35 - 36.

REFERENCES


1. G. Hadley: Linear Programming, Addison-Wesley, 1962.

2. G. Hadley: Linear Algebra, Addison-Wesley, 1961.

3. Saul I. Gass: Linear Programming, McGraw-Hill, 2nd edition, 1964.

4. Robert Dorfman: "Mathematical or 'Linear' Programming". A ER., Dec., 1953, pp. 797-825.

5. Robert Dorfman, Paul A. Samuelson, Robert M. Solow: Linear Programming and Economic Analysis, McGraw-Hill, 1958.

6. Samuel Karlin: Mathematical Methods and Theory in Games, Programming, and Economics, Vol. I, Addison-Wesley, 1959.

7. David Gale: The Theory of Linear Economic Models, McGraw-Hill, 1960.

8. G.B. Dantzig: "Maximization of Linear Function of Variables Subject to Linear Inequalities", Chapter 11 in T.C. Koopmans ed., Activity Analysis of Production and Allocation, John Wiley & Sons, 1951.

9. A. Charnes, W.W. Cooper, A. Henderson: Introduction to Linear Programming, John Wiley & Sons, 1953.

10. Kelvin Lancaster: Mathematical Economics, Macmillan, 1968.

11. Henri Theil, John C.G. Boot, Teun Kloek: Operations Research and Quantitative Economics, Translation from Dutch, McGraw-Hill, 1965.

12. J.G. Kemeny, H. Mirkil, J.L. Snell, and G.L. Thompson: Finite Mathematical Structures, Prentice-Hall, 1959.

13. R.G.D. Allen, Mathematical Economics, 2nd edition, Macmillan, 1959.

14. William J. Baumol, Economic Theory and Operations Analysis, 2nd edition, Prentice-Hall, 1965.